12 October 2009

Midterm Exam W 10/14 2 Hours

Closed Book, Closed Notes, No Calculators

1-D Fourier Transforms

$e^{-\pi x^2}$ } Gaussian

$e^{\pm i\pi x^2}$

$\mathrm{comb}(x)$ — $f(x)\cdot\frac{1}{\Delta x}\mathrm{comb}\left(\frac{x}{\Delta x}\right)$ Sampling

$$\int_{-\infty}^{+\infty} e^{-\pi x^2} e^{-2\pi i \xi x}\, dx$$

$$\int_{-\infty}^{+\infty} e^{-\pi\left(x^2 - 2i\xi x - \xi^2 + \xi^2\right)}\, dx$$

$$= \int_{-\infty}^{+\infty} e^{-\pi\xi^2}\, e^{-\pi\left(x^2 - 2i\xi x + (i\xi)^2\right)}\, dx$$

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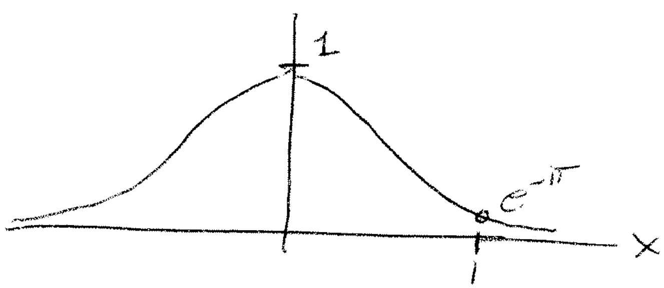
$$= e^{-\pi\xi^2} \underbrace{\int_{-\infty}^{+\infty} e^{-\pi(x - i\xi)^2}\, dx}_{\text{DEFINITE INTEGRAL} \;=\; 1}$$

$$\mathcal{F}_1\left\{e^{-\pi x^2}\right\} \propto e^{-\pi\xi^2}$$

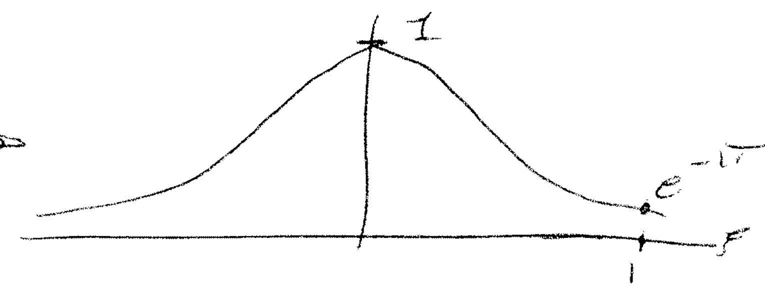
$$\mathcal{F}_1\left\{e^{-\pi x^2}\right\} = e^{-\pi\xi^2} \qquad \text{SELF-TRANSFORM}$$

$$\mathcal{F}_1\left\{e^{-\pi\left(\frac{x}{b_0}\right)^2}\right\} = |b_0|\, e^{-\pi(b_0\xi)^2} = |b_0|\, e^{-\pi\left(\frac{\xi}{b_0^{-1}}\right)^2}$$

IF $b_0 > 1 \Rightarrow b_0^{-1} < 1$



$\longrightarrow$



$$e^{\pm i\pi x^2}$$

$$\mathcal{F}_1\left\{e^{+i\pi x^2}\right\} \Rightarrow \mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\}$$

$$\int_{-\infty}^{+\infty} e^{+i\pi\left(\frac{x}{\alpha}\right)^2} e^{-2\pi i \xi x}\, dx$$

$$= \int e^{+i\pi\left(\left(\frac{x}{\alpha}\right)^2 - 2\xi x + (\alpha\xi)^2 - (\alpha\xi)^2\right)} dx$$

$$\underbrace{\left(\left(\frac{x}{\alpha}\right)^2 - 2\left(\frac{x}{\alpha}\right)\cdot(\alpha\xi) + (\alpha\xi)^2\right)}_{\left(\frac{x}{\alpha} - \alpha\xi\right)^2} - (\alpha\xi)^2$$

$$= \int_{-\infty}^{+\infty} e^{+i\pi\left(\frac{x}{\alpha} - \alpha\xi\right)^2} e^{-i\pi(\alpha\xi)^2}\, dx$$

$$\mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\} = \underline{e^{-i\pi(\alpha\xi)^2}} \, \underline{\int_{-\infty}^{+\infty} e^{+i\pi\left(\frac{x}{\alpha}-\alpha\xi\right)^2} dx}$$

NUMBER

$$\mathcal{F}_1\left\{e^{+i\pi\left(\frac{x}{\alpha}\right)^2}\right\} = e^{-i\pi\left(\frac{\xi}{\alpha^{-1}}\right)^2} \cdot \underline{\int_{-\infty}^{+\infty} e^{+i\pi\left(\frac{x}{\alpha}-\alpha\xi\right)^2} dx}$$

n.b.

α $\qquad$ α^{-1}

$$\int_{-\infty}^{+\infty} e^{+i\pi\left(\frac{x-\alpha^2\xi}{\alpha}\right)^2} dx$$

$x - \alpha^2\xi \equiv u \rightarrow$ TRANSLATION

$dx = du$

$x = \pm\infty \Rightarrow u = \pm\infty$

$$\int_{-\infty}^{+\infty} e^{+i\pi \left(\frac{u}{\alpha}\right)^2} du \Longrightarrow V \equiv \frac{u}{\alpha} \Rightarrow du = \alpha\, dv$$

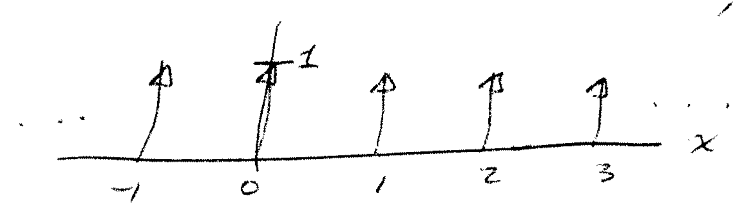
$$\int_{-\infty}^{+\infty} \alpha \, e^{+i\pi v^2} dv$$

$$|\alpha| \int_{-\infty}^{+\infty} e^{+i\pi v^2}\, dv = |\alpha|\, e^{+i\frac{\pi}{4}} = |\alpha|\, \frac{1+i}{\sqrt{2}}$$

$$\boxed{\mathcal{F}_1\left\{ e^{+i\pi\left(\frac{x}{\alpha}\right)^2} \right\} = |\alpha|\, e^{+i\frac{\pi}{4}}\, \underline{e^{-i\pi(\alpha\xi)^2}}}$$

$$\Rightarrow \mathcal{F}_1\left\{ \underbrace{e^{+i\pi x^2}}_{\text{IMPULSE RESPONSE IN FRESNEL DIFFRACTION REGION}} \right\} = \underbrace{e^{+i\frac{\pi}{4}}\, e^{-i\pi\xi^2}}_{\text{TRANSFER FUNCTION}}$$

$$\text{COMB}[x] = \sum_{n=-\infty}^{+\infty} \delta(x-n)$$



PERIODIC, DISCRETE, EVEN

EVEN $\Rightarrow \mathcal{F}_1\{\text{COMB}[x]\}$ IS

EVEN & REAL

$$\int_{-\infty}^{+\infty} \text{COMB}(x)\, e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} \delta(x-n) \right) e^{-2\pi i \xi x}\, dx$$

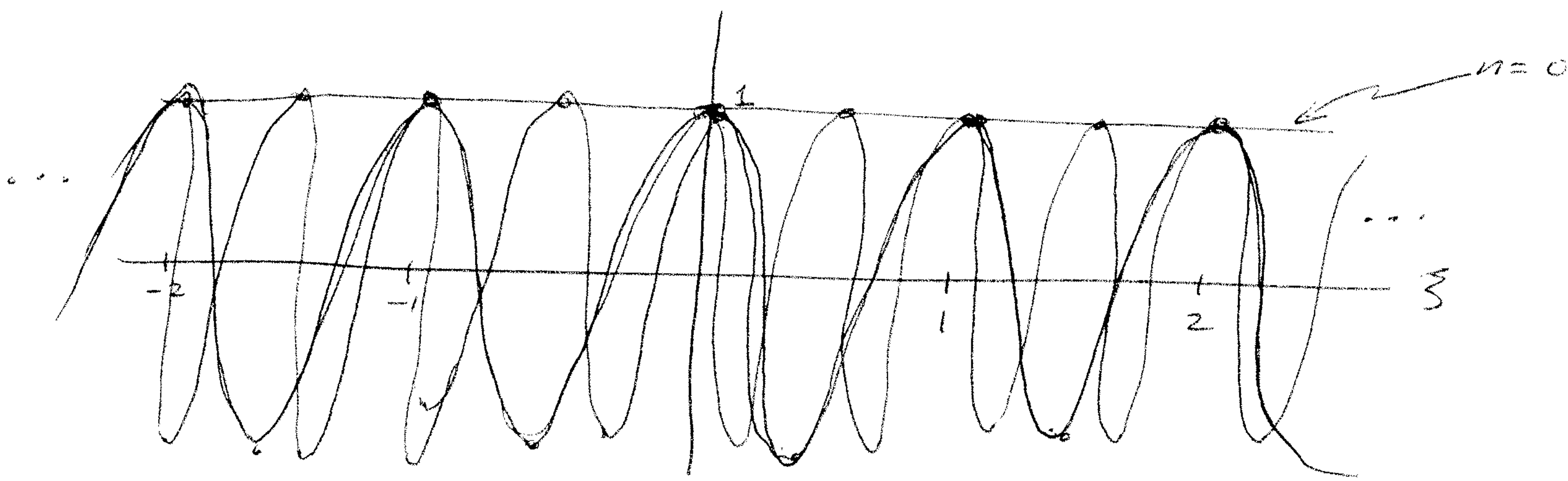
$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x-n)\, e^{-2\pi i \xi x}\, dx = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(x-n)\, \underline{e^{-2\pi i \xi \cdot n}}\, dx$$

VIA $\delta(x-x_0) \cdot f(x) = \delta(x-x_0)\, f(x_0)$

$$= \sum_n e^{-2\pi i \xi n} \underbrace{\int_{-\infty}^{+\infty} \delta(x-n)\, dx}_{1}$$

$$\mathcal{F}_1\{\text{COMB}(x)\} = \sum_{n=-\infty}^{+\infty} e^{-2\pi i \xi n}$$

$$\mathcal{F}_1\{\mathrm{comb}(x)\} = \sum_{n=-\infty}^{+\infty} e^{-2\pi i \xi n}$$



AREA OF $\mathcal{F}_1\{\mathrm{comb}(x)\}$ = AREA OF CONSTANT PART = ∞

$n=+1 \Rightarrow e^{-2\pi i \xi \cdot 1} = \cos(2\pi\xi) - i\sin(2\pi\xi)$

$n=-1 \Rightarrow e^{-2\pi i \xi \cdot (-1)} = \cos(2\pi\xi) + i\sin(2\pi\xi)$

$n=+2 \Rightarrow e^{-2\pi i \xi (+2)} = \cos(4\pi\xi) - i\sin(4\pi\xi)$

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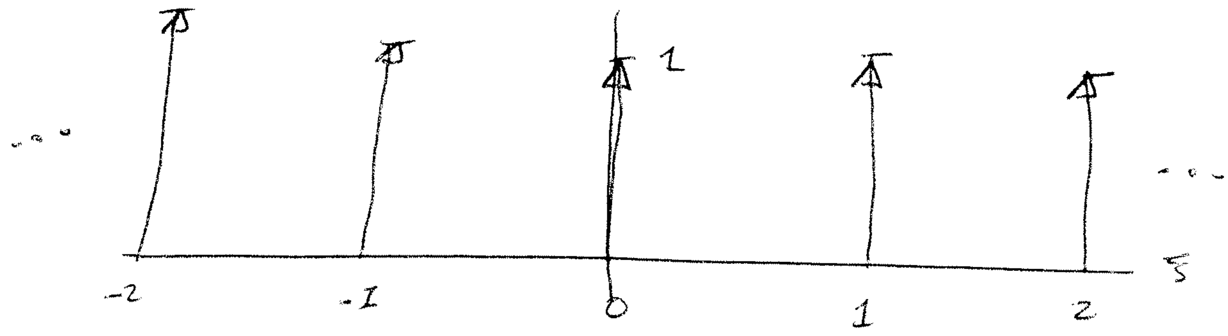


COMB(ξ)

$$\mathcal{F}_1\{\mathrm{comb}(x)\} = \mathrm{comb}(\xi) \qquad \text{SELF - TRANSFORM}$$

$$\mathcal{F}_1\{e^{-\pi x^2}\} = e^{-\pi\xi^2} \qquad \text{"}$$

$$\mathcal{F}_1\{e^{+i\pi \cancel{x}^2}\} = e^{+i\frac{\pi}{4}} e^{-i\pi\xi^2} \qquad \text{ALMOST}$$

$$\mathcal{F}\{e^{-i\frac{\pi}{8}} e^{+i\pi x^2}\} = e^{-i\frac{\pi}{8}} e^{+i\frac{\pi}{4}} e^{-i\pi\xi^2}$$

$$= e^{+i\frac{\pi}{8}} e^{-i\pi\xi^2} = e^{-i\pi(\xi^2 - \frac{1}{8})}$$

$$\mathcal{F}\{e^{+i\pi(x^2 - \frac{1}{8})}\}$$

$$\mathcal{F}\{\delta(x-x_0)\} = e^{-2\pi i x_0 \xi}$$

$$\Longrightarrow \mathcal{F}\{\delta(x)\} = 1(\xi)$$

$$\mathcal{F}\{\mathrm{RECT}(x)\} = \mathrm{SINC}(\xi)$$

$$\mathcal{F}\{\mathrm{SGN}(x)\} = \frac{1}{i\pi\xi} = 0(\xi) + i\left(-\frac{1}{\pi\xi}\right)$$

$$\mathcal{F}\{\mathrm{STEP}(x)\} = \mathcal{F}\left\{\frac{1}{2}\left(1(x) + \mathrm{SGN}(x)\right)\right\} = \frac{1}{2}\delta(\xi) - \frac{1}{2\pi i\xi}$$

$$\mathcal{F}\{e^{-x}\,\mathrm{STEP}(x)\} = \frac{1}{1+2\pi i\xi} = \frac{1-2\pi i\xi}{1+(2\pi\xi)^2}$$

$$\mathcal{F}\{e^{-\pi x^2}\} = e^{-\pi\xi^2} \quad (e^{-\pi\xi^2})$$

$$\mathcal{F}\{e^{+i\pi x^2}\} = e^{+i\frac{\pi}{4}}\, e^{-i\pi\xi^2} \Longrightarrow \mathcal{F}\{e^{-i\pi x^2}\} = e^{-i\frac{\pi}{4}} \cdot e^{+i\pi\xi^2}$$

$$\mathcal{F}\{\mathrm{COMB}(x)\} = \mathrm{COMB}(\xi)$$

THEOREMS OF F.T.

(1) MULTIPLICATION BY CONSTANT (LINEARITY OF INTEGRATION)

$$\mathcal{F}_1\{f[x]\} = F(\xi) \Longrightarrow \mathcal{F}_1\{\alpha f(x)\} = \alpha F(\xi)$$

$$\mathcal{F}_1\{0 \cdot f(x)\} = \mathcal{F}_1\{0(x)\} = 0(\xi)$$

(2) SUM OF TWO FUNCTIONS (LINEARITY OF INTEGRATION)

$$\mathcal{F}_1\{\alpha_1 f_1[x] + \alpha_2 f_2[x] + \cdots\} = \alpha_1 F_1[\xi] + \alpha_2 F_2[\xi] + \cdots$$

PROOFS OF THEOREMS

(1) CHANGE OF INTEGRATION VARIABLE

(2) SUBSTITUTE $\mathcal{F}_1^{-1}\{F(\xi)\}$ FOR $f(x)$; $f(x) = \int_{-\infty}^{+\infty} \underline{F(\xi)}\, \underline{\underline{e^{+2\pi i \xi x}}}\, d\xi$

FOURIER SYNTHESIS

(3) FOURIER TRANSFORM OF FOURIER TRANSFORM

IF $\mathcal{F}_1\{f(x)\} = F(\xi)$, WHAT IS $\mathcal{F}_1\{F(x)\}$

e.g., $\mathcal{F}_1\{\mathrm{Rect}(x)\} = \dfrac{\sin(\pi\xi)}{\pi\xi} \equiv \mathrm{Sinc}(\xi)$

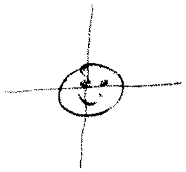
$$\mathcal{F}_1\{\mathrm{Sinc}(x)\} = \cancel{X}\ \ \mathrm{Rect}(\underline{\underline{-\xi}}) = \mathrm{Rect}(+\xi)$$

SPACE DOMAIN

$$\mathcal{F}_1\{F(x)\} = \int_{-\infty}^{+\infty} F(x)\, \underbrace{e^{-2\pi i \xi x}}\, dx$$

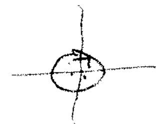
$$\mathcal{F}_1^{-1}\{F(\xi)\} = \int_{-\infty}^{+\infty} F(\xi)\, e^{+2\pi i \xi x}\, d\xi$$

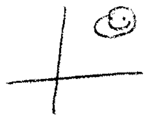
$$\mathcal{F}_1\{F(x)\} = \int_{-\infty}^{+\infty} F(x)\, e^{+2\pi i(-\xi)\cdot x}\, dx = f(-\xi)$$

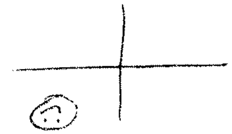


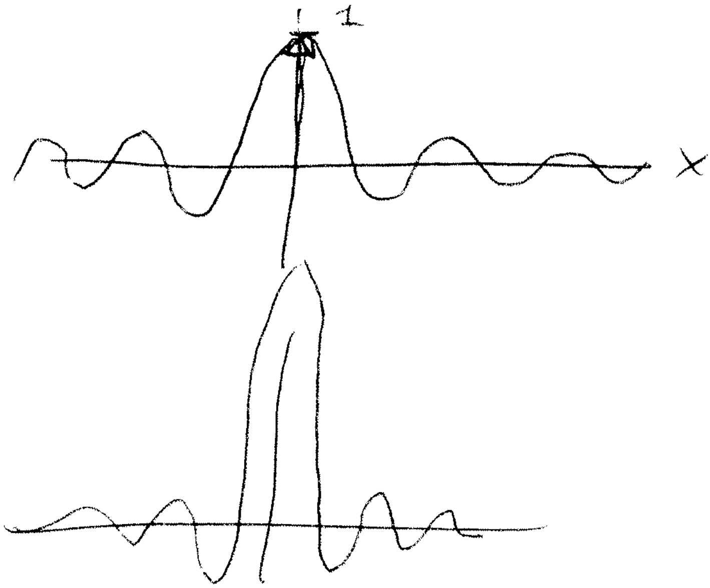
$f(x,y) \longrightarrow F(\xi,\eta)$

$F(x,y) \longrightarrow f(-\xi,-\eta)$

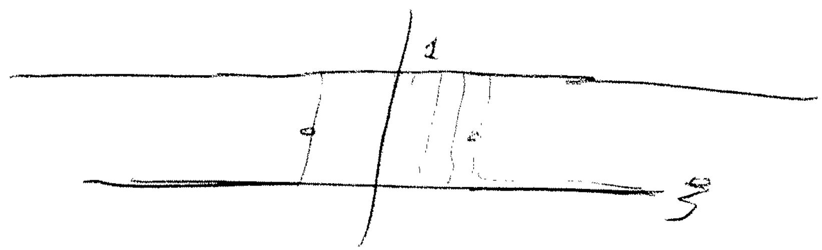








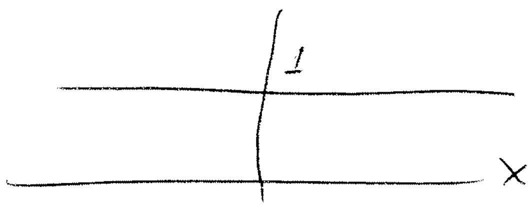
$\xrightarrow{\mathcal{F}_1}$

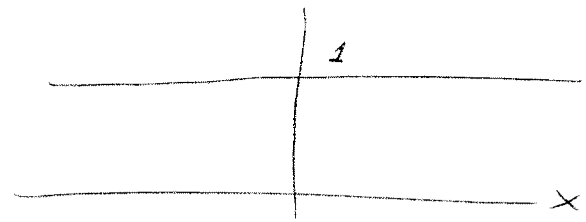


SINC(x) IS LOWPASS-FILTERED VERSION OF $\delta(x)$

$$\mathcal{F}\{\delta(x)\} = 1(\xi)$$

$$\mathcal{F}\{1(x)\} = \delta(-\xi) = \delta(+\xi)$$



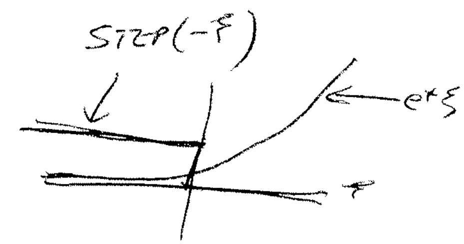


$$\delta(x-x_0) \longrightarrow \underline{1(\xi)\cdot e^{-2\pi i \xi x_0}} = \cos(2\pi\xi x_0) - i\sin(2\pi\xi x_0)$$

$$1(x)\, e^{-2\pi i \xi_0 x} \longrightarrow \delta(-\xi-\xi_0) = \delta(-(\xi+\xi_0)) = \delta(\xi+\xi_0)$$

$$e^{-x}\,\mathrm{STEP}(x) \longrightarrow \frac{1}{1+2\pi i\xi}$$

$$\frac{1}{1+2\pi i x} \longrightarrow e^{-(-\xi)}\,\mathrm{STEP}(-\xi) = e^{+\xi}\,\mathrm{STEP}(-\xi)$$

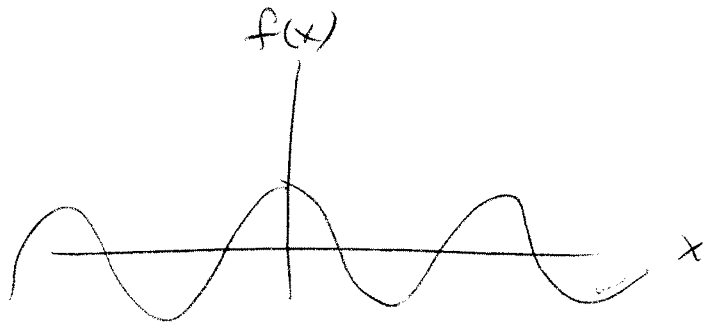


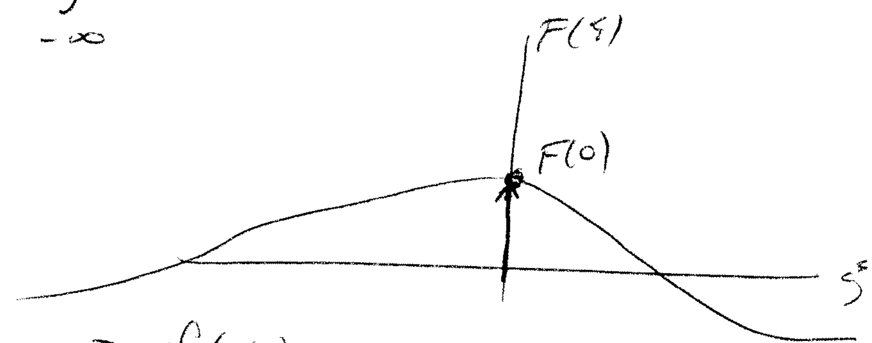
CENTRAL ORDINATE THEOREM

$$\mathcal{F}\{f(x)\} = F(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

$$\Rightarrow \underline{F[0]} = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \cdot 0 \cdot x}\, dx$$

$$= \int_{-\infty}^{+\infty} f(x)\, dx = \underline{\text{AREA OF } f(x)}$$





CENTRAL ORDINATE = AREA OF f(x)
OF F(ξ)

$$\mathcal{F}_1^{-1}\{F(\xi)\} = \int_{-\infty}^{+\infty} F(\xi)\, e^{+2\pi i \xi x}\, d\xi = f(x)$$

$$f(x=0) = \int_{-\infty}^{+\infty} F(\xi)\, d\xi = \text{AREA OF } F(\xi)$$

$$\mathcal{F}_1\{e^{-\pi x^2}\} = e^{-\pi \xi^2} \Rightarrow \int_{-\infty}^{+\infty} e^{-\pi x^2} dx = e^{-\pi \cdot 0^2} = 1$$

$$\mathcal{F}_1\{e^{\pm i\pi x^2}\} = e^{\pm i\frac{\pi}{4}}\, e^{\mp i\pi\xi^2}$$

AREA OF $e^{\pm i\pi x^2} = e^{\pm i\frac{\pi}{4}}$

SCALING THEOREM

IF $\mathcal{F}_1\{f(x)\} = F(\xi)$, WHAT IS $\mathcal{F}_1\{f(\frac{x}{b_0})\}$?

b_0 IS REAL AND $b_0 \neq 0$

IF $b_0 > 1 \Rightarrow$ WIDER FUNCTION

IF $0 < b_0 < 1 \Rightarrow$ NARROWER FN.

IF $b_0 < -1 \Rightarrow$ WIDER & REVERSED

$-1 < b_0 < 0 \Rightarrow$ NARROWER & REVERSED

$b_0 > 1$

$$\int_{-\infty}^{+\infty} f\left[\frac{x}{b_0}\right] e^{-2\pi i \xi x}\, dx$$

$u \equiv \frac{x}{b_0} \Rightarrow dx = b_0\, du$

$x = \pm\infty \Rightarrow u = \pm\infty$

$$\int_{-\infty}^{+\infty} f[u]\, e^{-2\pi i \xi \overbrace{(u b_0)}^{x}}\, b_0\, du = b_0 \underbrace{\int_{-\infty}^{+\infty} f[u]\, e^{-2\pi i (b_0 \xi) u}\, du}$$

$$\mathcal{F}\left\{f\left[\frac{x}{b_0}\right]\right\} \text{ For } b_0 > 0 = b_0 \, \mathcal{F}_1\left\{f(x)\right\}\Big|_{\xi \to b_0 \xi}$$

$$\mathcal{F}_1\left\{ f\left(\frac{x}{b_0}\right)\right\} = b_0 \, F\left[b_0 \xi\right] \quad \text{IF} \quad b_0 > 0$$

$$= b_0 \, F\left[\frac{\xi}{b_0^{-1}}\right] = b_0 \cdot F\left[\frac{\xi}{1/b_0}\right]$$

$$\text{IF } b_0 > 1 \Rightarrow 0 < \frac{1}{b_0} < 1$$

$$\text{IF } b_0 < 0 \Rightarrow \int_{-\infty}^{+\infty} f\left[\frac{x}{b_0}\right] e^{-2\pi i \xi x} \, dx \quad ; \quad u \equiv \frac{x}{b_0} \Rightarrow dx = +|b_0| \, du$$

$$x = +\infty \Rightarrow u = -\infty$$

$$x = -\infty \Rightarrow u = +\infty$$

$$\int_{u=+\infty}^{u=-\infty} f[u] \, e^{-2\pi i \xi \cdot b_0 \cdot u} \, |b_0| \, du$$

$$|b_0| \, F(b_0 \xi)$$

$$\mathcal{F}_1\left\{f\left(\frac{x}{b_0}\right)\right\} = |b_0|\, F[b_0 \cdot \xi] = |b_0| \cdot F\left[\frac{\xi}{1/b_0}\right]$$

IF $b_0 = -1 \Rightarrow \mathcal{F}_1\{f[-x]\} = |-1| \cdot F\left[\frac{\xi}{-1}\right]$

$$\boxed{\mathcal{F}_1\{f[-x]\} = F[-\xi]}$$

RECT(x) $\longrightarrow$

RECT(x) $\longrightarrow$ SINC(ξ)

RECT$\left[\frac{x}{2}\right] \longrightarrow$ 2 SINC(2ξ) = 2 SINC$\left(\frac{\xi}{1/2}\right)$

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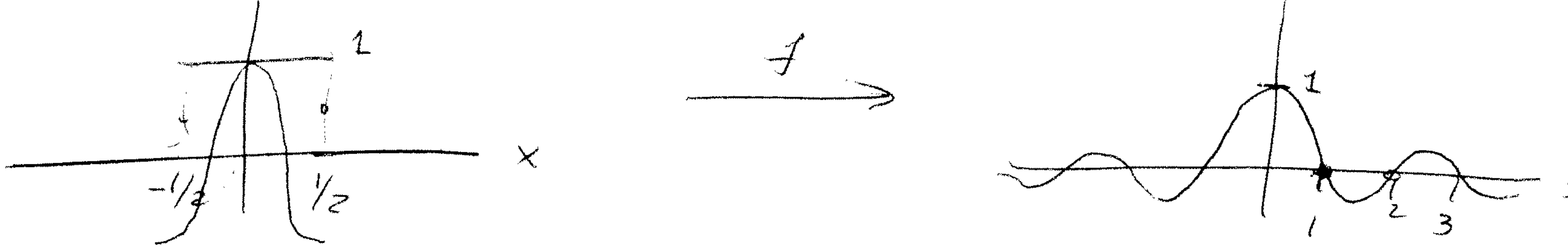
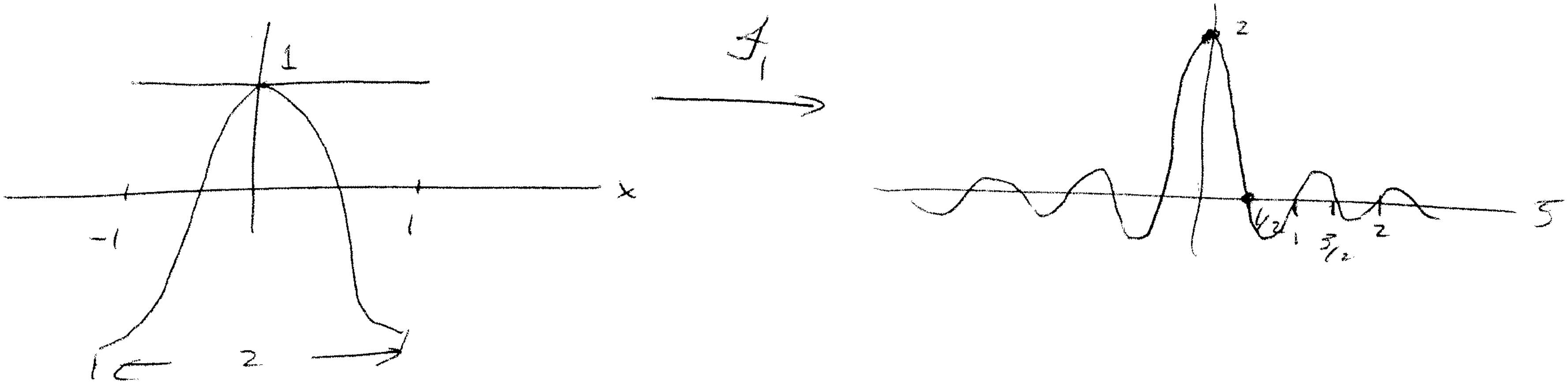


$\mathrm{Rect}(x) \perp \cos(2\pi \cdot 1 \cdot x)$



SHIFT THEOREM

IF $f(x) \xrightarrow{\mathcal{F}_1} F(\xi)$, THEN $f(x-x_0) \xrightarrow{\mathcal{F}_1}$?

$$\mathcal{F}_1\{f(x-x_0)\} = \int_{-\infty}^{+\infty} f(x-x_0)\, e^{-2\pi i \xi x}\, dx$$

$u \equiv x - x_0 \Rightarrow x = u + x_0$

$du = dx$

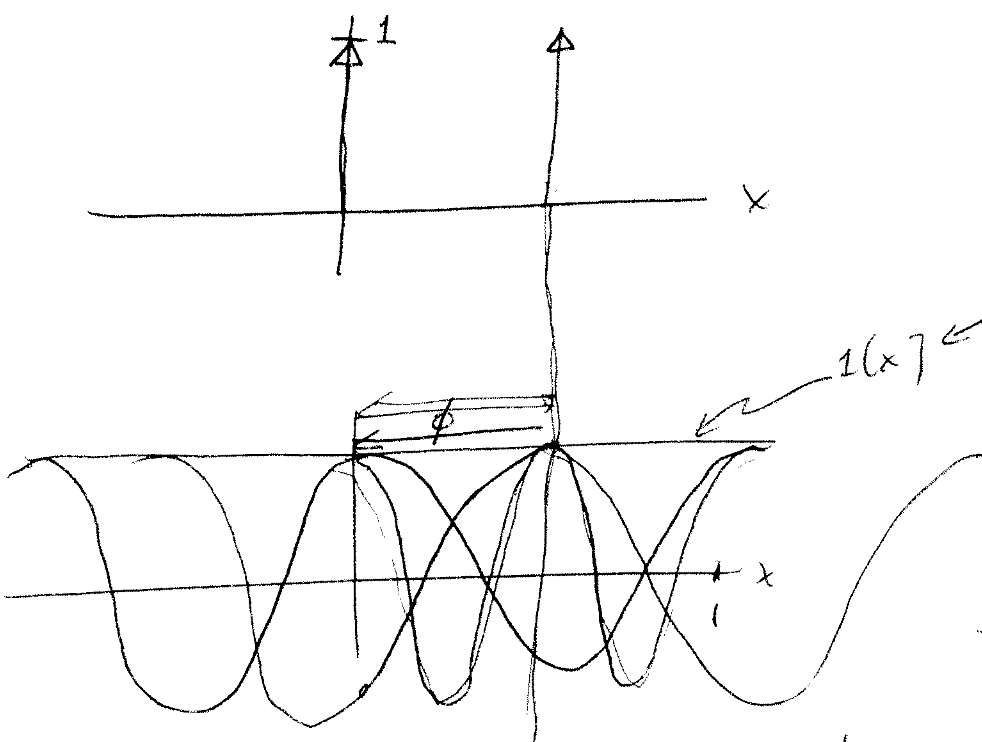
& $x = \pm\infty \Rightarrow u = \pm\infty$

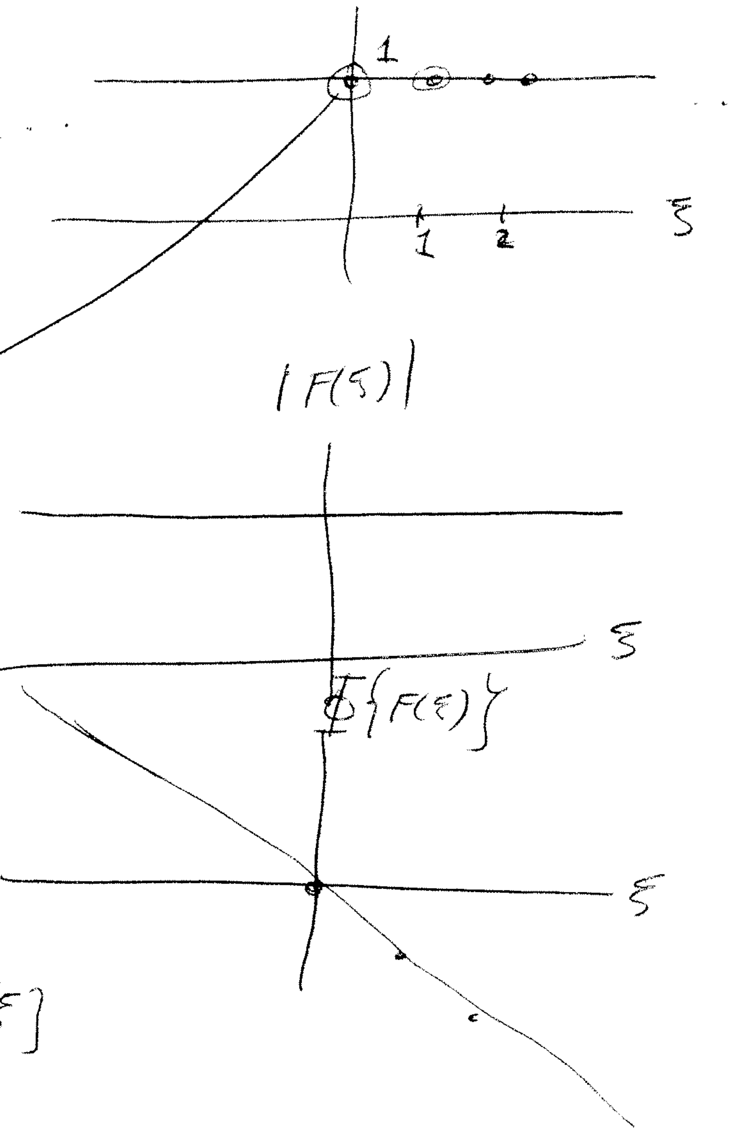
$$\int_{u=-\infty}^{u=+\infty} f(u)\, e^{-2\pi i \xi (u+x_0)}\, du$$

$$= \int_{u=-\infty}^{+\infty} f(u)\, e^{-2\pi i \xi u}\, e^{-2\pi i \xi x_0}\, du = e^{-2\pi i \xi x_0} \underbrace{\int_{-\infty}^{+\infty} f(u)\, e^{-2\pi i \xi u}\, du}_{F(\xi)}$$

$$\boxed{\mathcal{F}_1\{f(x-x_0)\} = e^{-2\pi i \xi x_0}\, F(\xi)}$$ SV

$$f[x] = \delta(x) \Rightarrow F(\xi) = 1[\xi]$$





$$\boxed{\mathcal{F}_1\{f(x-x_0)\} = \underline{\underline{e}}^{-2\pi i \xi x_0} F[\xi]}$$

$$= \left[\cos(2\pi\xi x_0) - i\sin(2\pi\xi x_0)\right] F(\xi]$$