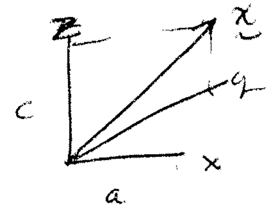
OCTOBER 7

MIDTERM NEXT WEDNESDAY

FOURIER ANALYSIS $\Rightarrow$ DETERMINING SINUSOIDAL COMPONENTS OF $f(x)$



$$\underset{\sim}{x} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

$F(\xi)$ $\qquad$ $f(x)$

$$\int_{-\infty}^{+\infty} f(x)\left(e^{+2\pi i \xi x}\right)^{*} dx = F(\xi)$$

ANALYSIS (DECOMPOSITION)

$$\boxed{F(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \xi x}\, dx}$$

LSV

CANNOT IMPLEMENT AS CONVOLUTION

$M - C - M$, $\underbrace{C - M - C}_{\text{OPTICAL SYSTEM}}$

$$f[x] = f_R[x] + i\, f_I[x]$$

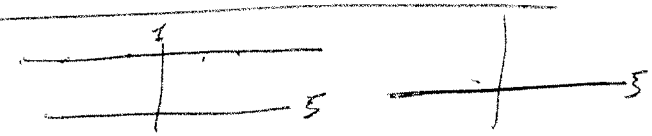
$$\mathcal{F}_1\{f(x)\} = \int_{-\infty}^{+\infty} \left[f_R[x] + i f_I(x) \right] \left[\cos[2\pi\xi x] - i \sin[2\pi\xi x] \right] dx$$

$$= \int \left[\left[\underbrace{f_R[x] \cdot \cos(2\pi\xi x)}_{\text{EVEN PART OF } f_R} + \underbrace{f_I(x) \cdot \sin(2\pi\xi x)}_{\text{ODD PART OF } f_I} \right] + i \left[\underbrace{f_I[x] \cos(2\pi\xi x)}_{\text{EVEN PART OF } f_I} - \underbrace{f_R(x) \sin(\,)}_{\text{ODD OF } f_R} \right] \right] dx$$

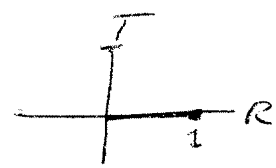
(RESPONDS TO EVEN PART OF f_R)

EXAMPLES

$$\delta[x] \longrightarrow 1[\xi]$$



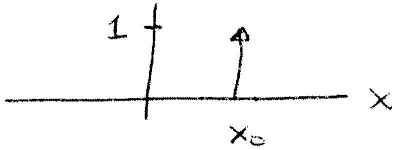
$$\delta[x] + i\, 0(x) \longrightarrow 1(\xi] + i \cdot 0[\xi]$$



$$|F(\xi)| = 1[\xi]$$

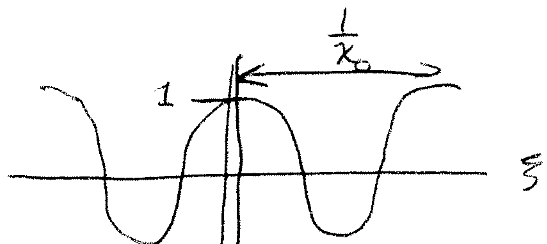
$$\mathcal{I}\{F(\xi)\} = 0[\xi]$$

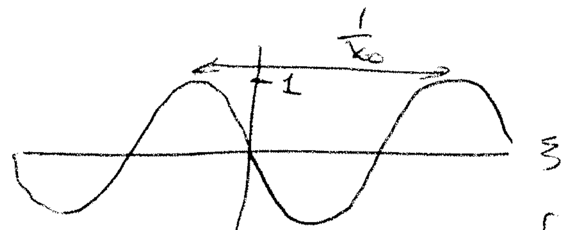
$$\delta[x-x_0] \xrightarrow{\mathcal{F}_1} \int_{-\infty}^{+\infty} \underbrace{\delta(x-x_0]\, e^{-2\pi i \xi x}} dx$$

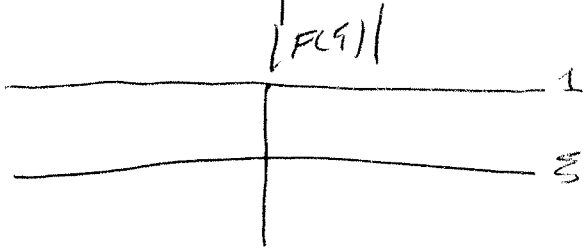


$$\int_{-\infty}^{+\infty} \delta(x-x_0]\, e^{-2\pi i \xi x_0}\, dx$$

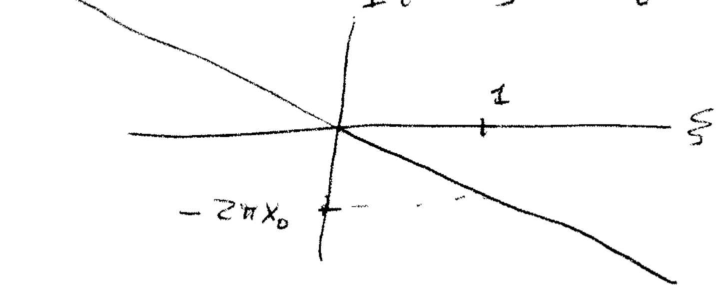
$$= \int_{-\infty}^{+\infty} e^{-2\pi i \xi x_0} \int_{-\infty}^{+\infty} \delta(x-x_0) dx = e^{-2\pi i \xi x_0} = F[\xi]$$



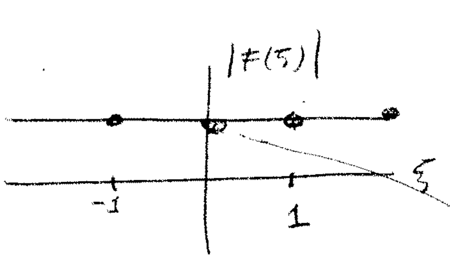


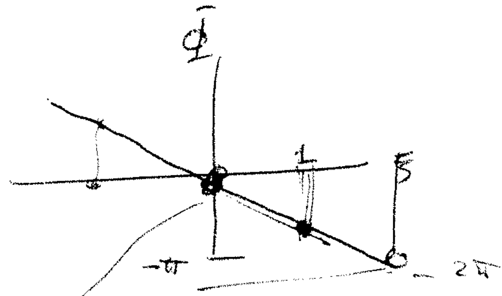


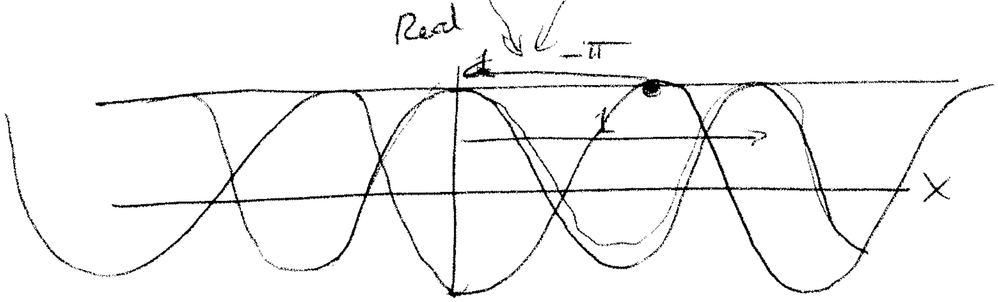
$$\Phi\{F(\xi)\} = \tan^{-1}\left[\frac{-\sin}{\cos}\right] = -2\pi \xi x_0$$

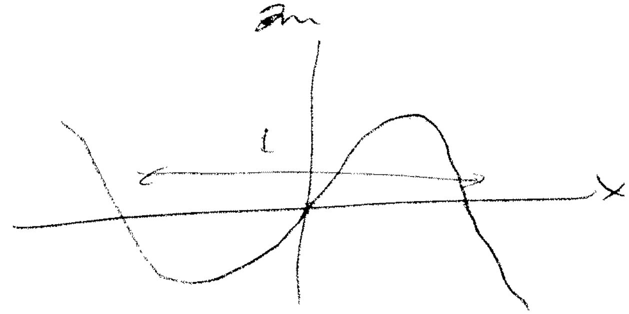


(SHIFT THEOREM)



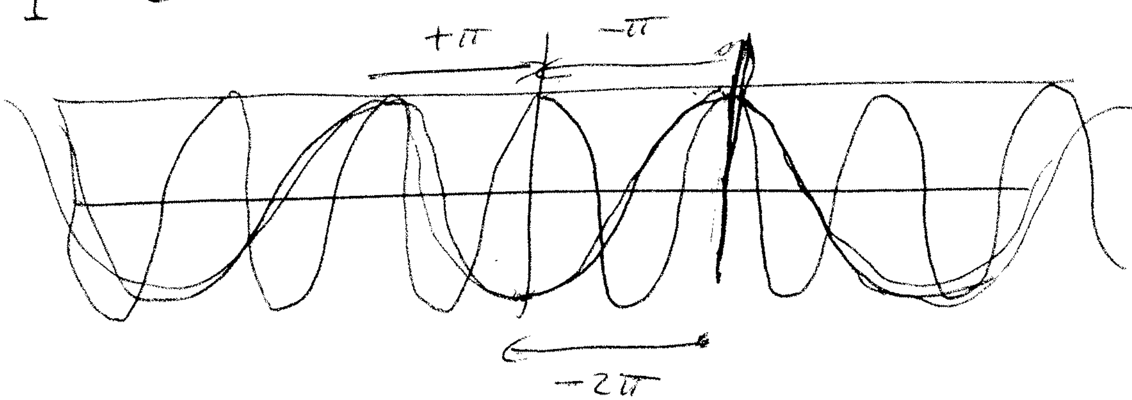


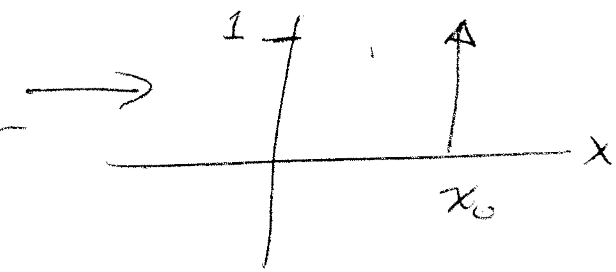




$$1 \quad e^{+2\pi i\cdot 0\cdot x} = \cos(0) + i\sin(0) = 1[x] + i\,0[x]$$

$$1 \quad e^{+2\pi i\cdot 1\cdot x} = \cos(2\pi x) + i\sin(2\pi x)$$





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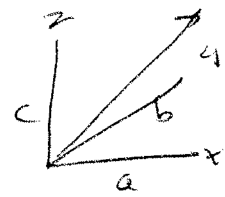
$$f[x] \underset{\mathcal{F}_1^{-1}}{\overset{\mathcal{F}_1}{\rightleftarrows}} F[\xi]$$



$$\underbrace{F[\xi_1]}_{a}\,\underbrace{e^{+2\pi i \xi_1 x}}_{\text{VECTOR COMPONENTS}} + \underbrace{F[\xi_2]}_{b}\,\underbrace{e^{+2\pi i \xi_2 x}}_{\text{VECTOR COMPONENT}} + \cdots = \underbrace{f(x)}_{\underline{x}\ \text{VECTOR}}$$

$$\boxed{f[x] = \int_{-\infty}^{+\infty} F(\xi)\cdot e^{+2\pi i \xi x}\, d\xi}$$

FOURIER SYNTHESIS

$$\cancel{\xi} \quad \delta(x) = \cancel{e} \int_{-\infty}^{+\infty} e^{+2\pi i \xi x}\, d\xi$$

$$\delta(x - x_0) = \int_{-\infty}^{+\infty} e^{+2\pi i \xi (x - x_0)}\, d\xi$$

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$$\delta(x-x_0) = \int_{-\infty}^{+\infty} e^{+2\pi i \xi (x-x_0)}\, d\xi$$

$$f[x] = \int_{-\infty}^{+\infty} f[\alpha]\, \delta(\overset{x-\alpha}{\cancel{\alpha - x}})\, d\alpha$$

$$= \int_{-\infty}^{+\infty} f(\alpha) \left(\int_{-\infty}^{+\infty} e^{+2\pi i \xi (\overset{x_0-\alpha}{\cancel{\alpha - x_0}})}\, d\xi \right) d\alpha$$

$$= \int_{-\infty}^{+\infty} f(\alpha) \int e^{-2\pi i \xi \alpha}\, e^{+2\pi i \xi x_0}\, d\xi\, d\alpha$$

$$= \int \left(\int f(\alpha)\, e^{-2\pi i \xi \alpha}\, d\alpha \right) e^{+2\pi i \xi x_0}\, d\xi$$

$$= \int F(\xi)\, e^{+2\pi i \xi x_0}\, d\xi = f(x_0)$$

$$\mathcal{F}_1\{f(x)\} = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \xi x}\, dx \equiv F(\xi) \qquad \text{ANALYSIS}$$

$$\mathcal{F}_1^{-1}\{F(\xi)\} = \int_{-\infty}^{+\infty} F(\xi)\, e^{+2\pi i \xi x}\, \underline{d\xi} = f(x) \qquad \text{SYNTHESIS}$$

$$\delta[x] \longrightarrow 1[\xi] \Rightarrow \mathcal{F}_1^{-1}\{1(\xi)\} = \delta(x)$$

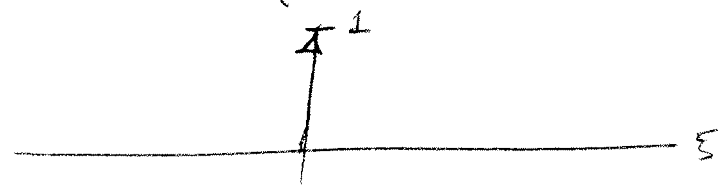
$$1[\cancel{x}] \xrightarrow{?} \delta(\xi] \qquad \mathcal{F}_1^{-1}\{\delta(\xi)\}$$

$$\int_{-\infty}^{+\infty} \delta(\xi)\, e^{+2\pi i \xi x}\, d\xi = \int_{-\infty}^{+\infty} \delta(\xi - 0)\, e^{+2\pi i \cdot 0 \cdot x}\, d\xi$$

$$= e^{+2\pi i \cdot 0 \cdot x} \int_{-\infty}^{+\infty} \delta(\xi)\, d\xi = 1[x] \cdot 1$$

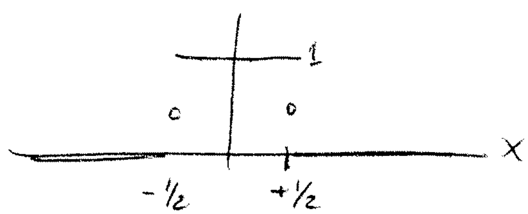
$\mathcal{F}\{1(x)\} = \delta(\xi)$

$\underline{\underline{1(x)}}$



EXAMPLE

$$\mathrm{Rect}(x) \equiv \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$



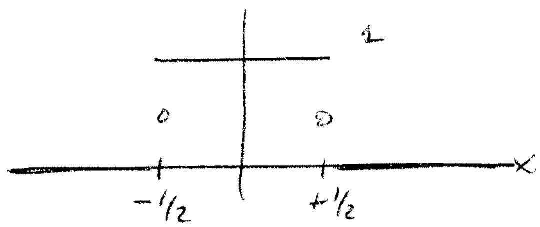
$$\int_{-\infty}^{+\infty} \mathrm{Rect}(x)\, e^{-2\pi i \xi x}\, dx = \int_{-1/2}^{+1/2} e^{-2\pi i \xi x}\, dx$$

$$\int_b^c e^{Ax}\, dx = \frac{1}{A} e^{Ax} \Big|_{x=b}^{x=c} \Longrightarrow \frac{1}{-2\pi i \xi} e^{-2\pi i \xi x} \Big|_{x=-1/2}^{+1/2}$$

$$= \frac{e^{-2\pi i \xi \cdot \frac{1}{2}} - e^{-2\pi i \xi \left(-\frac{1}{2}\right)}}{-2\pi i \xi} = \frac{-e^{+i\pi\xi} + e^{-i\pi\xi}}{-2i} \cdot \frac{1}{\pi\xi}$$

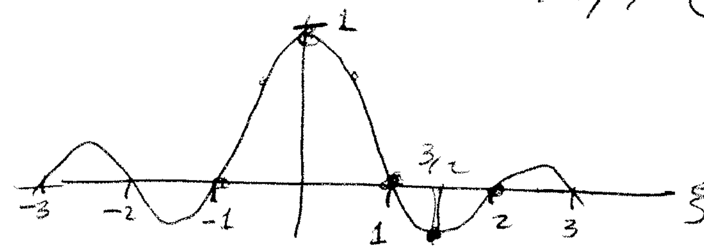
$$= \frac{\sin(\pi\xi)}{\pi\xi} \equiv \mathrm{sinc}(\xi) \qquad \boxed{\mathrm{Rect}(x) \longrightarrow \mathrm{Sinc}(\xi)}$$

$\mathrm{Rect}[x] + i \cdot 0[x]$



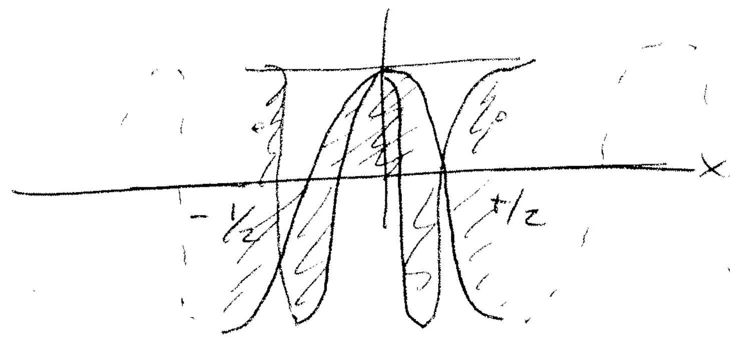
$\mathrm{Rect}[x] + i \cdot 0[x]$

$\longrightarrow$

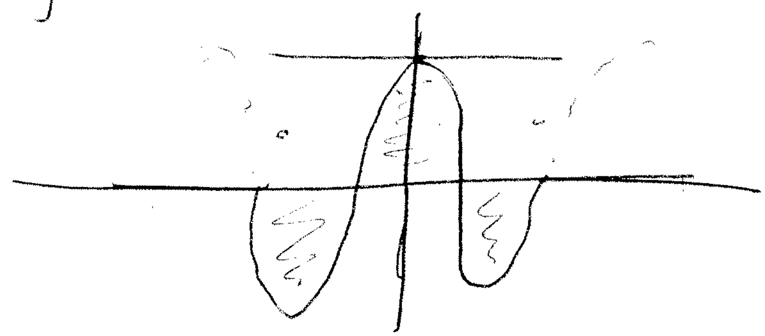


$\mathrm{Sinc}(\xi) + i \cdot 0(\xi)$

$\mathrm{Rect}(x) \perp \cos[2\pi \cdot 1 \cdot x]$



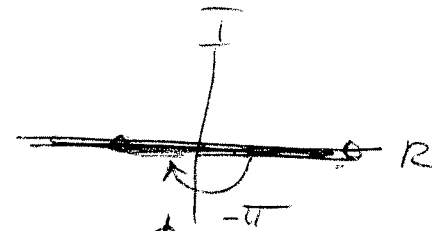
$\mathrm{Rect}(x) \quad \cos\left(2\pi \cdot \frac{3}{2} \cdot x\right]$

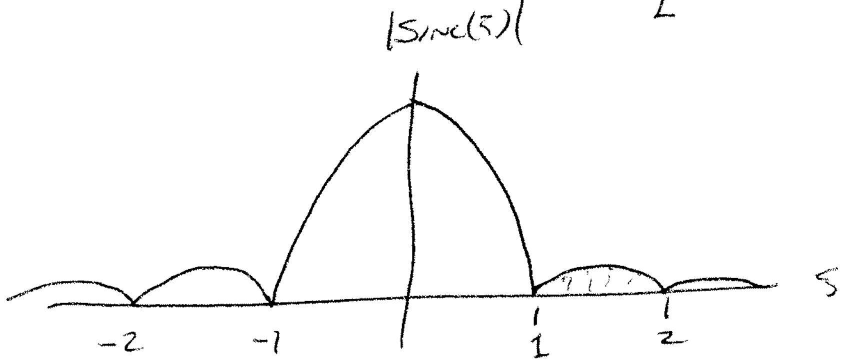


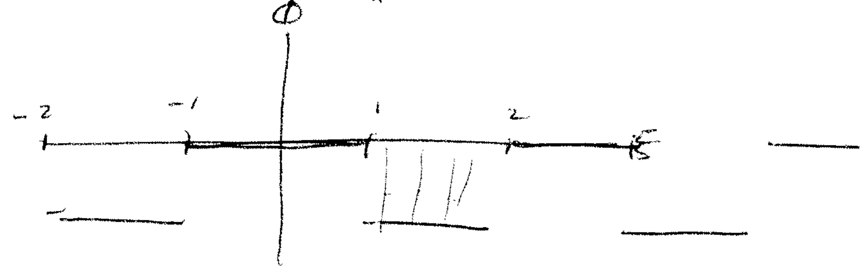
$$F(\xi) = \mathrm{sinc}(\xi) + i \cdot 0(\xi)$$

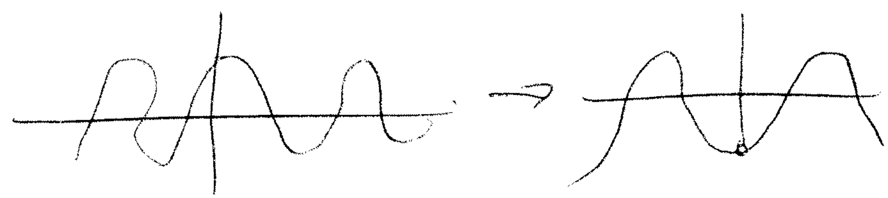
$$|F(\xi)| = \sqrt{\mathrm{sinc}^2(\xi) + (0(\xi))^2} = |\mathrm{sinc}(\xi)|$$

$$\Phi\{F(\xi)\} = \tan^{-1}\left[\frac{0(\xi)}{\mathrm{sinc}(\xi)}\right]$$







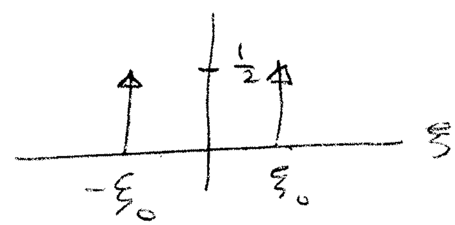


$$1\cos(2\pi\xi_0 x) \xrightarrow{\ \mathcal{F}_1\ } \int_{-\infty}^{+\infty} \cos(2\pi\xi_0 x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{+\infty} \cos(2\pi\xi_0 x)\left[\cos(2\pi\xi x) - i\sin(2\pi\xi x)\right] dx$$

$$= \int_{-\infty}^{+\infty} \cos[2\pi\xi_0 x]\cos[2\pi\xi x]\, dx - i\cancelto{0}{\int_{-\infty}^{+\infty} \cos(2\pi\xi_0 x)\sin(2\pi\xi x)\, dx}$$

$$= \frac{1}{2}\delta[\xi+\xi_0] + \frac{1}{2}\delta(\xi-\xi_0]$$



SYNTHESIS

$$\int_{-\infty}^{+\infty} \frac{1}{2}\delta[\xi+\xi_0]\cdot e^{+2\pi i\xi x}\, d\xi + \int_{-\infty}^{+\infty} \frac{1}{2}\delta(\xi-\xi_0)\, e^{+2\pi i \xi x}\, d\xi$$

$$\underline{\frac{1}{2}e^{-2\pi i\xi_0 x}}\ \cancelto{1}{\int_{-\infty}^{+\infty}\delta(\xi+\xi_0]\, d\xi} + \underline{\frac{1}{2}e^{+2\pi i\xi_0 x}}\ \cancelto{1}{\int_{-\infty}^{+\infty}\delta(\xi-\xi_0)\, d\xi}$$

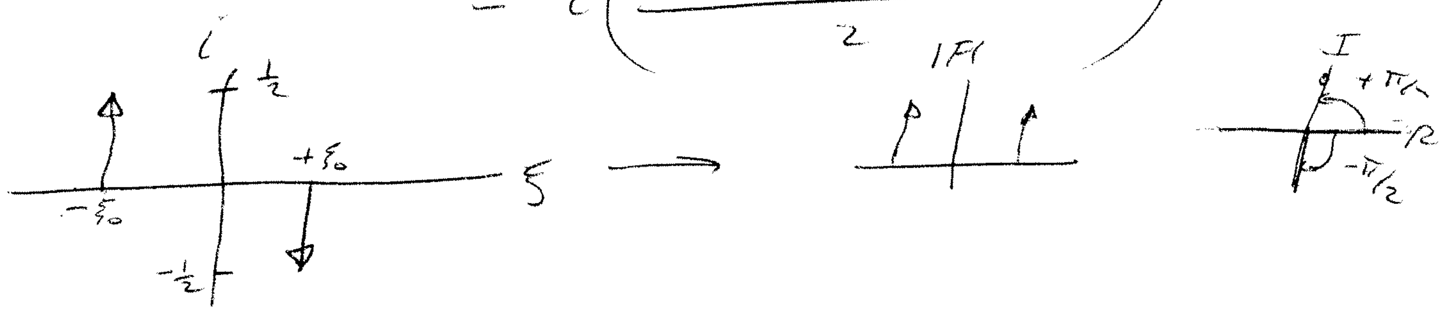
$$\frac{1}{2}e^{-2\pi i\xi_0 x} + \frac{1}{2}e^{+2\pi i\xi_0 x} = \cos(2\pi\xi_0 x)$$

$$\mathcal{F}_1\{\sin(2\pi\xi_0 x)\}$$

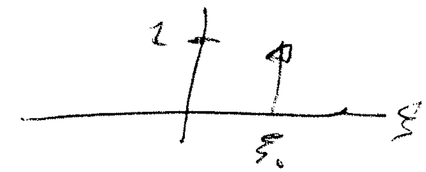
$$\frac{e^{+i(2\pi\xi_0 x)} - e^{-i(2\pi\xi_0 x)}}{2i} \longrightarrow \frac{\delta(\xi-\xi_0) - \delta(\xi+\xi_0)}{2i}$$

$$\frac{1}{i} = -i \qquad -\frac{i}{2}\left[\delta(\xi-\xi_0) - \delta(\xi+\xi_0)\right]$$

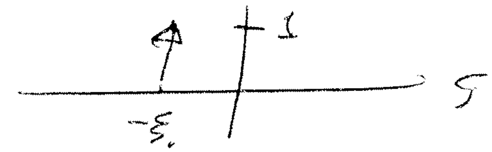
$$= i\left(\frac{\delta(\xi+\xi_0) - \delta(\xi-\xi_0)}{2}\right)$$

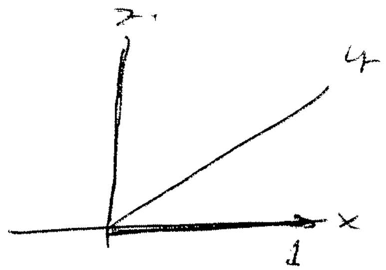


$$\mathcal{F}_1\left\{ e^{+i(2\pi \xi_0 x)} \right\} \longrightarrow \delta(\xi - \xi_0)$$



$$\mathcal{F}_1\left\{ e^{-i(2\pi \xi_0 x)} \right\} \longrightarrow \delta(\xi + \xi_0)$$

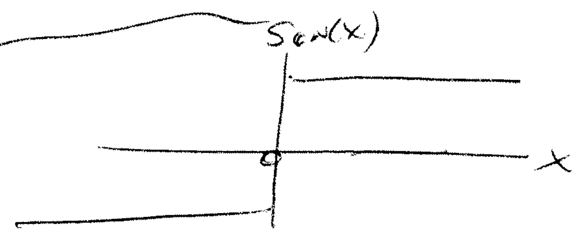




$$\underset{\sim}{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$f(x)$	$F(\xi)$
$\delta(x) + i\,0(x)$	$1(\xi) + i\,0(\xi)$
$1(x)$	$\delta(\xi)$
$\delta(x-x_0)$ $\longrightarrow$	$e^{-2\pi i \xi x_0} \cdot 1(\xi)$
$\delta(x+x_0)$ $\longrightarrow$	$e^{+2\pi i \xi x_0} \cdot 1(\xi)$
$\cos(2\pi \xi_0 x)$ $\longrightarrow$	$\cancel{\tfrac{1}{2}\left(e^{+2\pi i \xi_0}\right.}$ $\tfrac{1}{2}\,\delta(\xi+\xi_0) + \tfrac{1}{2}\,\delta(\xi-\xi_0)$
$e^{+2\pi i \xi_0 x}$ $\longrightarrow$	$\delta(\xi-\xi_0)$
$e^{-2\pi i \xi_0 x}$ $\longrightarrow$	$\delta(\xi+\xi_0)$
$\mathrm{Rect}(x)$ $\longrightarrow$	$\mathrm{Sinc}(\xi)$

$$\mathrm{SGN}(x) \equiv \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



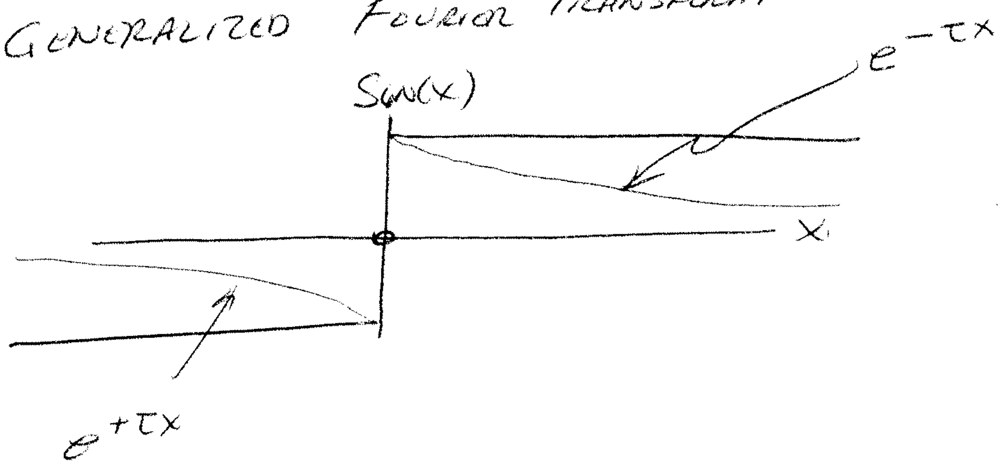
ODD + REAL $\longrightarrow$ ODD + IMAGINARY

$$\int_{-\infty}^{+\infty} \mathrm{SGN}(x)\, e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{0} (-1)\, e^{-2\pi i \xi x}\, dx + \int_{0}^{\infty} (+1)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{0}^{\infty} e^{-2\pi i \xi x}\, dx - \int_{-\infty}^{0} e^{-2\pi i \xi x}\, dx \qquad ?$$

~~A~~

GENERALIZED FOURIER TRANSFORM $\Rightarrow$ LIMIT



$$\lim_{\tau \to 0} e^{-\tau x} = 1$$

$$\mathrm{SGN}(x) \longrightarrow \lim_{\tau\to 0} e^{-\tau|x|}\,\mathrm{SGN}(x)$$

$$\int_{-\infty}^{+\infty} \lim_{\tau\to 0} e^{-\tau|x|}\,\mathrm{SGN}(x)\, e^{-2\pi i\xi x}\,dx$$

$$= \lim_{\tau\to 0}\left[\int_{-\infty}^{0} (-1)\, e^{+\tau x} e^{-2\pi i\xi x}\,dx + \int_{0}^{\infty} (+1)\, e^{-\tau x} e^{-2\pi i\xi x}\,dx\right.$$

$$= \lim_{\tau\to 0}\left(\int_{-\infty}^{0} - e^{(\tau-2\pi i\xi)x}\,dx + \int_{0}^{\infty} e^{-(\tau+2\pi i\xi)x}\,dx\right]$$

$$\int_b^c e^{Ax}\,dx = \frac{1}{A} e^{Ax}\Big|_{x=b}^{x=c}$$

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$$\lim_{\tau\to 0}\left\{ -\left.\frac{e^{(\tau-2\pi i\xi)x}}{\tau-2\pi i\xi}\right|_{x=-\infty}^{x=0} + \left.\frac{e^{-(\tau+2\pi i\xi)x}}{-(\tau+2\pi i\xi)}\right|_{x=0}^{x=\infty}\right\}$$

$$\lim_{\tau\to 0}\left\{ -\frac{1-0}{\tau-2\pi i\xi} + \frac{0-1}{-(\tau+2\pi i\xi)}\right\}$$

$$\lim_{\tau\to 0}\left[\cancel{\frac{2}{\tau+}} \quad -\frac{1}{\tau-2\pi i\xi} + \frac{1}{+\tau+2\pi i\xi}\right]$$

$$\longrightarrow \quad +\frac{1}{2\pi i\xi} + \frac{1}{2\pi i\xi} = \frac{2}{2\pi i\xi} = \frac{1}{i\pi\xi}$$

$$\mathcal{F}\{\operatorname{sgn}(x)\} = 0[\xi] + i\left(-\frac{1}{\pi\xi}\right)$$

$$\frac{1}{i} = -i$$

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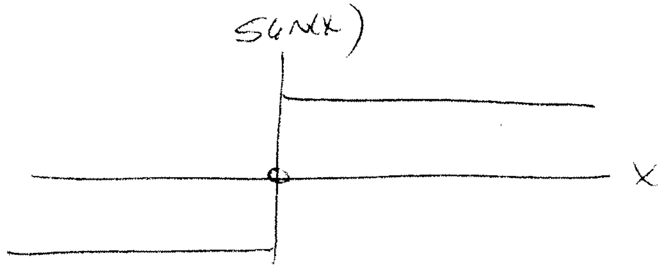
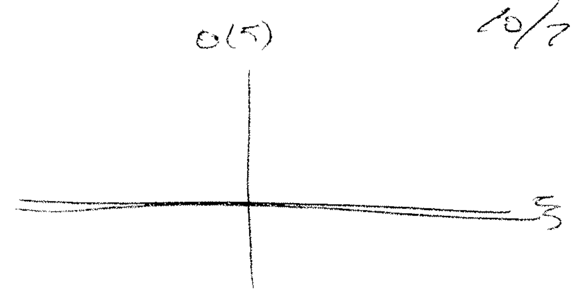
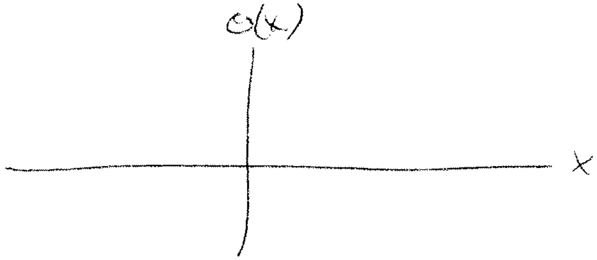
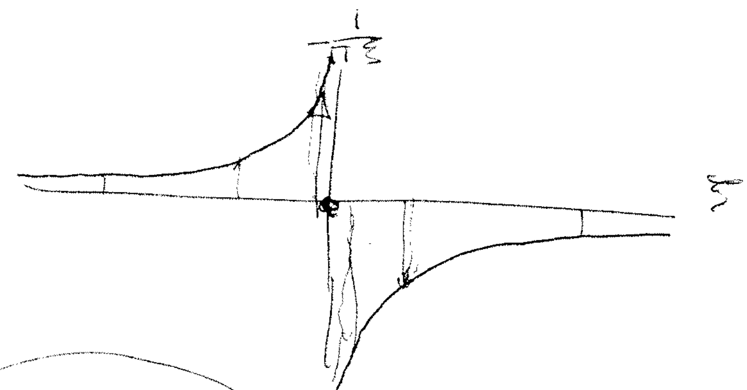
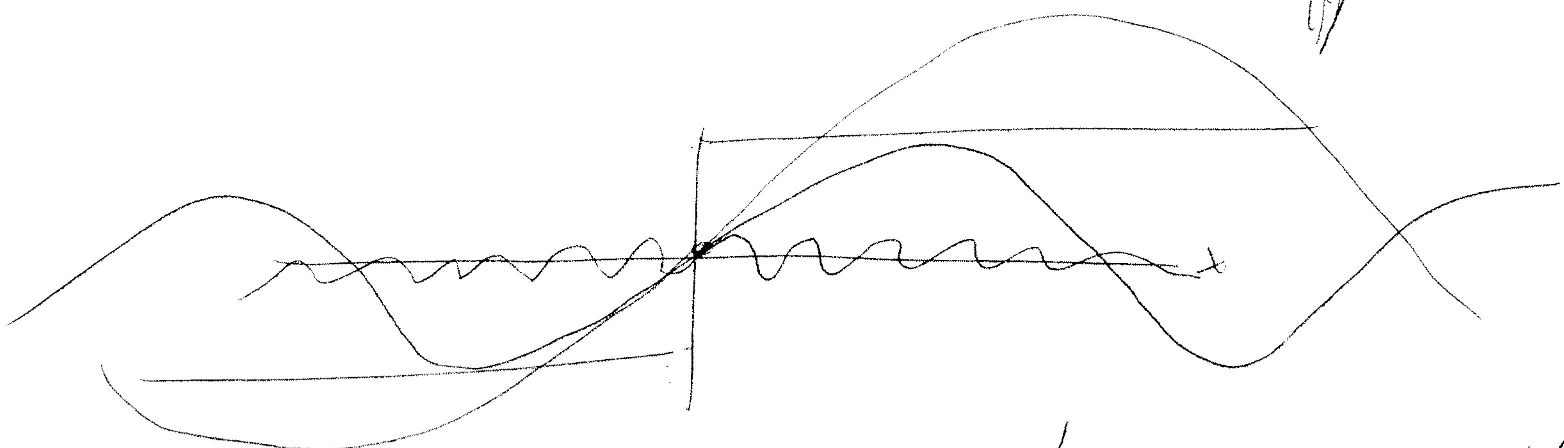


$\xrightarrow{\mathcal{F}}$







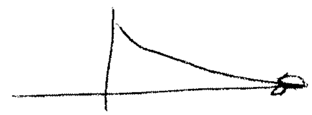


$$SGN(x) \longrightarrow \underline{\underline{\frac{1}{i\pi\xi}}} \rightarrow 0(\xi) + i\left(\frac{-1}{\pi\xi}\right)$$

$$\mathcal{F}\{\mathrm{SGN}(x)\} = \int_0^{\infty} \underline{\underline{e^{-2\pi i \xi x}}}\, dx - \int_{-\infty}^{0} e^{+2\pi i \xi x}\, dx$$

$$\frac{e^{-2\pi i \xi x}}{-2\pi i \xi}$$

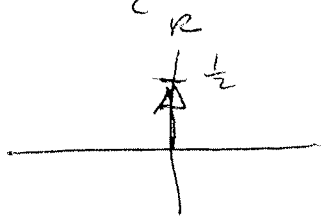
$$e^{(\tau \pm 2\pi i \xi)} \rightarrow$$

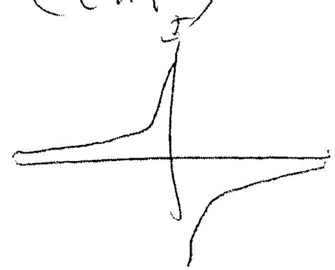


e

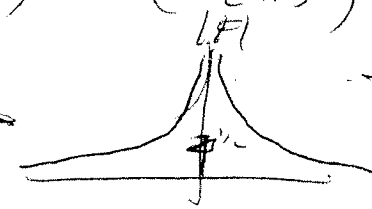
$$\mathrm{STEP}[x] = \underbrace{\tfrac{1}{2}\, 1[x]}_{\text{even}} + \underbrace{\tfrac{1}{2}\, \mathrm{SGN}(x)}_{\text{ODD}}$$

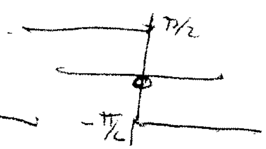
$$\mathcal{F}\{\mathrm{STEP}(x)\} = \frac{1}{2}\delta(\xi) + \frac{1}{2}\left(\frac{1}{i\pi\xi}\right) = \frac{1}{2}\delta(\xi) + i\left(-\frac{1}{2\pi\xi}\right)$$





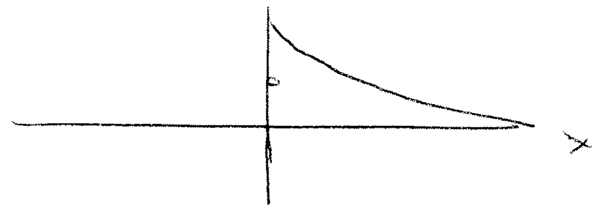
$\rightarrow$





Φ

$e^{-x}\,\mathrm{STEP}(x)$



$$\mathcal{F}\{e^{-x}\,\mathrm{STEP}(x)\} = \int_{-\infty}^{+\infty} e^{-x}\,\underline{\mathrm{STEP}(x)}\;e^{-2\pi i\xi x}\,dx$$

$$= \int_0^\infty e^{-x}\,e^{-2\pi i\xi x}\,dx$$

$$= \int_0^\infty e^{-(1+2\pi i\xi)x}\,dx$$

$$= \left.\frac{e^{-(1+2\pi i\xi)x}}{-(1+2\pi i\xi)}\right|_{x=0}^{x=\infty} - \frac{0-1}{(1+2\pi i\xi)}$$

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$$\mathcal{F}\{e^{-x}\,\mathrm{step}[x]\} = \frac{1}{1+2\pi i\xi}$$

$$= \frac{1}{(1+2\pi i\xi)}\cdot\left(\frac{1-2\pi i\xi}{1-2\pi i\xi}\right) = \frac{1-2\pi i\xi}{1+(2\pi\xi)^2}$$

$$= \frac{1}{1+(2\pi\xi)^2} + i\left(\frac{-2\pi\xi}{1+(2\pi\xi)^2}\right)$$