10/5/09

HOMEWORK #4 NOW POSTED

DUE M 10/12

MIDTERM W 10/14 2 HOURS

CONVOLUTION – LSI OPERATION

$$f[x] * h[x] \equiv \int_{-\infty}^{+\infty} f[\alpha]\, h[x-\alpha]\, d\alpha = h[x] * f[x]$$

CONVOLUTION COMPLETELY SPECIFIED BY psf $h[x]$

$$f[x,y] * h[x,y] = \iint_{-\infty}^{+\infty} f[\alpha,\beta]\, h[x-\alpha, y-\beta]\, d\alpha\, d\beta$$

VARIANT OF CONVOLUTION – CORRELATION

VECTOR $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \longrightarrow |\underset{\sim}{x}| = \sqrt{\sum_{n=1}^{N} |x_n|^2}$

$\underset{\sim}{x} \longrightarrow f(x)$

$\uparrow$ COORDINATE OF f ANALOGOUS TO COMPONENT INDEXED BY n

$$|x|^2 \longrightarrow \int_{-\infty}^{+\infty} f(\alpha) \cdot f(\alpha)\, d\alpha \longrightarrow \sum_{n=1}^{N} |x_n|^2 \geq 0$$

$$\int_{-\infty}^{+\infty} |f(\alpha)|^2\, d\alpha$$

ASIDE IF $\int_{-\infty}^{+\infty} |f(\alpha)|^2 d\alpha = 1$

IF $\underset{\sim}{x}$ HAS COMPLEX COORDINATES $x_n \longrightarrow x_n^2$; $|x_n|^2 = x_n$

$$|x_n|^2 = x_n x_n^* = x_n^* x_n$$

ANALOGY FOR $f(x) = \underbrace{\int_{-\infty}^{+\infty} f(\alpha) \cdot f^*(\alpha)\, d\alpha}$ or $\int_{-\infty}^{+\infty} f^*(\alpha) f(\alpha)\, d\alpha$

$$\int_{-\infty}^{+\infty} \underbrace{f[\alpha]}_{\text{INPUT}} \underbrace{f^*[\alpha]}_{\text{REFERENCE}} d\alpha = \text{REAL NUMBER}$$

TRANSLATE REFERENCE TO COORDINATE x

$$\int_{-\infty}^{+\infty} f[\alpha]\, f^*[\alpha - x]\, d\alpha = \text{COMPLEX NUMBER VARY WITH TRANSLATION } x$$

$$= g[x]$$

REWRITE AS CONVOLUTION

$$\int_{-\infty}^{+\infty} f[\alpha]\, f^*[-x+\alpha]\, d\alpha = \int f(\alpha)\, f^*(-(x-\alpha))\, d\alpha$$

$$g(x) = f(x) * f^*(-x)$$

$$g(x) = \boxed{f(x) \star f[x] \equiv f(x) * f^*(-x)}$$

$$f[x] \star f[x]\Big|_{x=0} = \int_{-\infty}^{+\infty} f(\alpha)\, f^*(\alpha - 0)\, d\alpha = \int_{-\infty}^{\infty} |f(\alpha)|^2 d\alpha \geq 0$$

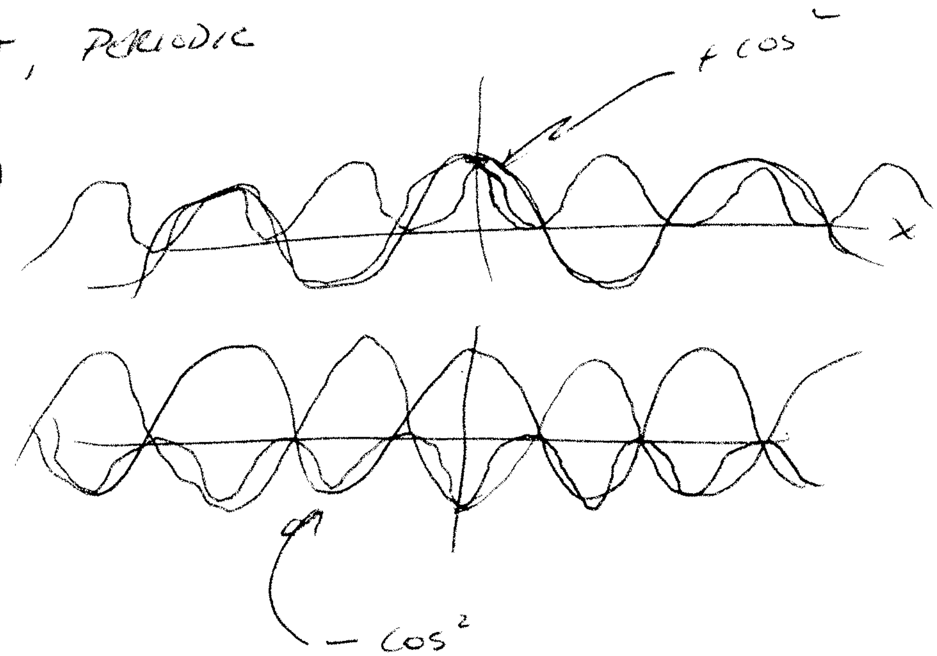
$$f(x) \star f(x) = f(x) * f^*(-x) \Rightarrow \text{AUTOCORRELATION}$$

(1) MAXIMUM IS AT $x = 0 \Rightarrow$ WHERE $f(x)$ IS MOST SIMILAR TO SELF

(2) MEASURE OF "SELF-SIMILARITY" FOR DIFFERENT TRANSLATIONS

IF $f(x)$ HAS INFINITE SUPPORT, PERIODIC

$$f(x) = \cos(2\pi \xi_0 x)$$



$$\lim_{B \to \infty} \int_{-B}^{+B} f(\alpha) f^*(\alpha - x)\, d\alpha$$

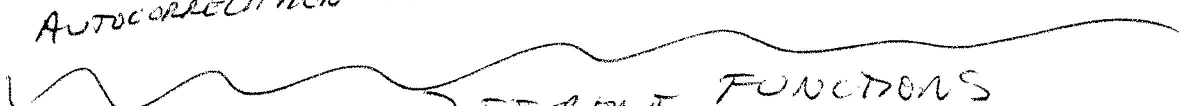
NORMALIZED AUTOCORRELATION

$$\frac{f[x] \star f[x]}{\left(f[x] \star f[x] \Big|_{x=0} \right)} \equiv \gamma_{ff}$$

REAL PART IS EVEN, IMAGINARY PART IS ODD

γ_{ff} IS HERMITIAN

AUTOCORRELATION IS OFTEN USED TO CHARACTERIZE STOCHASTIC FUNCTIONS



GENERALIZE TO DIFFERENT FUNCTIONS

INPUT $f(x)$
REFERENCE $m[x]$

$\Big\}$ $f[x] \star m[x]$ MEASURES SIMILARITY OF $f(x)$ TO $m[x]$ FOR DIFFERENT SHIFTS

$$f(x) \star m(x) = f[x] \ast m^*[-x]$$

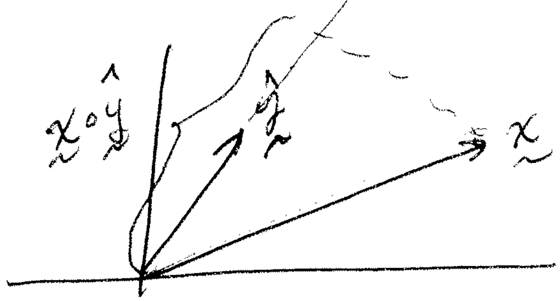
$$= \int_{-\infty}^{+\infty} f(\alpha)\, m^*[\alpha - x]\, d\alpha$$

CROSSCORRELATION – ~~MOR~~ LARGE FOR TRANSLATIONS x WHERE $f(x)$ LOOKS LIKE $m[x]$

$$\underset{\sim}{x} \circ \underset{\sim}{y} = \sum_{n=1}^{N} x_n y_n \rightarrow \sum_{n=1}^{N} x_n y_n^*$$

IF $|\underset{\sim}{y}| = 1 \rightarrow$ "PROJECTION" OF $\underset{\sim}{x}$ ONTO DIRECTION OF $\underset{\sim}{y}$

$$|\hat{\underset{\sim}{y}}| \equiv 1$$



PROJECTING $f(x]$ ONTO "DIRECTION" OF $m(x]$

CROSSCORRELATION FINDS "MATCH" BETWEEN $f(x)$ AND TRANSLATED REFERENCE FUNCTION

IF $\left. f(x) \star m(x) \right|_{x=x_0} = 0$ $\quad f(x)$ & $m(x)$ ARE ORTHOGONAL

$\underset{\sim}{x} \cdot \underset{\sim}{y} = 0 \Rightarrow \underset{\sim}{x} \perp \underset{\sim}{y}$

GENERALIZE CROSS CORRELATION

INPUT $f(x)$

REFERENCE $m(x)$

$$\int_{-\infty}^{+\infty} f(\alpha)\, \underbrace{m^{*}(\alpha - x)}_{\text{TRANSLATING}}\, d\alpha = g(x)$$

PARAMETER

$$\int f(\alpha)\, m^{*}(\alpha; x)\, d\alpha$$

PARAMETER

PROJECTING $f(\alpha)$ ONTO $m^{*}(\alpha; x)$

CHANGE NAME OF x

$$\int_{-\infty}^{+\infty} f(\alpha)\, m^{*}(\alpha; \xi)\, d\alpha$$

PARAMETER

FOR EXAMPLE $m(\alpha; \xi) = e^{+2\pi i \xi \alpha}$

$m(x; \xi) = e^{+2\pi i \xi x}$

$$\int_{-\infty}^{+\infty} f(\alpha) \left(e^{+2\pi i \xi \alpha} \right)^{*} d\alpha \rightarrow \text{PROJECTING } f(x) \text{ ONTO } \underline{e^{+2\pi i \xi x}}$$

9

$$\int_{-\infty}^{+\infty} f(\alpha)\left(e^{+2\pi i \xi \alpha}\right)^{*} d\alpha = \int_{-\infty}^{+\infty} f(\alpha)\, e^{-2\pi i \xi \alpha}\, d\alpha = g[\xi]$$

$$\equiv F[\xi]$$ (DIFFERENT COORDINATE SYSTEM)

LSV OPERATION $\mathcal{O}\{f(x)\} = g(x) \rightarrow F(\xi)$

$$\mathcal{O}\{f(x-x_0)\} \neq F(\xi - x_0)$$

$$f(x) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{O}\to\mathcal{F}}{\rightleftarrows}} F(\xi)$$ EQUIVALENT REPRESENTATION OF $f(x)$
AND SOMETIMES MORE CONVENIENT

$$\int_{-\infty}^{+\infty} f(\alpha)\, m^{*}(\alpha; u)\, d\alpha \qquad m(\alpha, u) = \cancel{\delta(\ )}\ \delta(\alpha - u)$$

$$\int f(\alpha)\, \delta^{*}(\alpha - u)\, d\alpha = f(u)$$ BECAUSE $\delta^{*}(u) = \delta(u)$

$$f(x) = A_0 \cos(2\pi \overset{\text{FIXED}}{\xi_0} x)$$ (10)

$$m(x;\xi) = \cos(2\pi \underset{\text{VARIABLE}}{\xi} x)$$

$$\int_{-\infty}^{+\infty} A_0 \cos(2\pi \xi_0 \alpha) \left(\cos(2\pi \xi \alpha)\right)^{*} d\alpha$$

n.b.

$$= A_0 \int_{-\infty}^{+\infty} \cos(2\pi \xi_0 \alpha) \cos(2\pi \xi \alpha)\, d\alpha$$

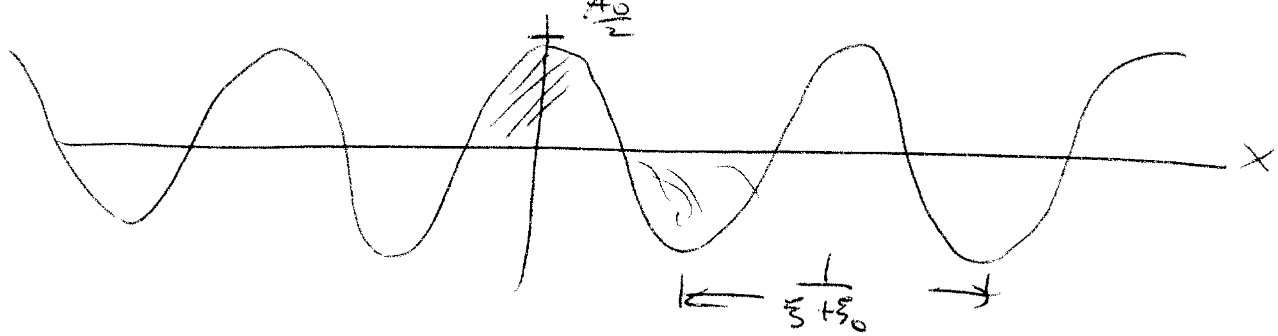
$$\cos A \cos B = \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\rightarrow \frac{A_0}{2} \int_{-\infty}^{+\infty} \cos\left(2\pi(\xi+\xi_0)\alpha\right) d\alpha + \frac{A_0}{2} \int_{-\infty}^{+\infty} \cos\left(2\pi(\xi-\xi_0)\alpha\right) d\alpha$$

(11)

$$\frac{A_0}{2}\int_{-\infty}^{+\infty} \cos\left(2\pi\underbrace{(\xi+\xi_0)}\,x\right)\,dx \Rightarrow \frac{A_0}{2}\,\delta(\xi+\xi_0)$$

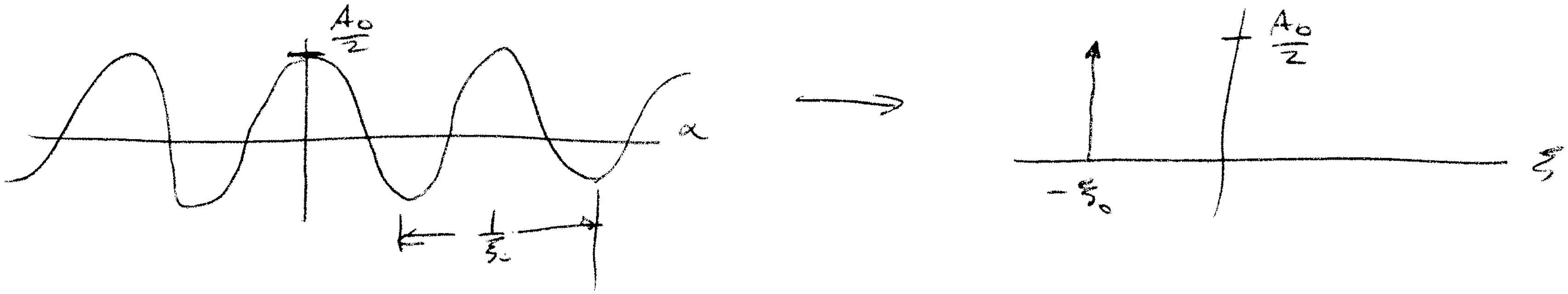


WHAT IF $\xi = -\xi_0$? $\quad \xi+\xi_0 = 0 \Rightarrow \frac{A_0}{2}\int_{-\infty}^{+\infty} \cos(0)\,dx = \frac{A_0}{2}\int_{-\infty}^{+\infty} 1\,dx = \infty$

$$\int_{-\infty}^{+\infty} e^{\pm 2\pi i \xi x}\,d\xi = \delta(x) \rightarrow \cancel{\int_{-\infty}^{+\infty} \cos(2\pi(\xi+\xi_0)x\,dx)}$$

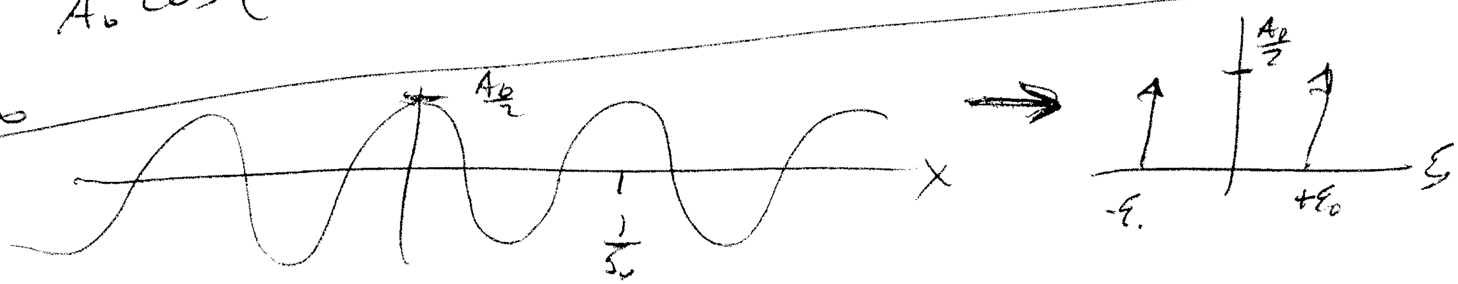
$$\int_{-\infty}^{+\infty} \cos\left(2\pi(\xi+\xi_0)x\right)dx$$

$$\frac{A_0}{2}\int_{-\infty}^{+\infty} \cos\left(2\pi(\xi+\xi_0)\alpha\right) d\alpha = \frac{A_0}{2}\,\delta(\xi+\xi_0)$$



$$\frac{A_0}{2}\int_{-\infty}^{+\infty} \cos\left(2\pi(\xi-\xi_0)\alpha\right) d\alpha = \frac{A_0}{2}\,\delta(\xi-\xi_0)$$

$$\boxed{\int_{-\infty}^{+\infty} A_0\cos(2\pi\xi_0\alpha)\,\cos(2\pi\xi\alpha)\,d\alpha = \frac{A_0}{2}\,\delta(\xi+\xi_0) + \frac{A_0}{2}\,\delta(\xi-\xi_0)}$$



$$\underbrace{A_0 \cos(2\pi \xi_0 x)}_{f(x)} = \underbrace{\frac{A_0}{2}\left(e^{+2\pi i \xi_0 x} + e^{-2\pi i \xi_0 x} \right)}$$

$$\frac{A_0}{2}\left(\delta[\xi - \xi_0] + \delta(\xi + \xi_0) \right)$$

EVEN INPUT, EVEN REFERENCE

EVEN INPUT , ODD REFERENCE

$$\left.\begin{aligned} f(x) &= A_0 \cos(2\pi \xi_0 x) \\ m(x) &= \sin(2\pi \xi x) \end{aligned}\right\} \int_{-\infty}^{+\infty} A_0 \cos(2\pi \xi_0 x) \sin(2\pi \xi x)\, dx$$

$$\sin(A \pm B) = \sin A \cos B \pm \underline{\cos A \sin B}$$

$$\cos A \sin B = \frac{1}{2} \sin(A+B) - \frac{1}{2} \sin(A-B)$$

$$\frac{A_0}{2}\int_{-\infty}^{+\infty} \sin(2\pi(\xi + \xi_0)x)\, dx - \frac{A_0}{2}\int_{-\infty}^{+\infty} \sin(2\pi(\xi - \xi_0)x)\, dx$$

$$\xi = -\xi_0 \Rightarrow \frac{A_0}{2}\int \sin(0)\, dx = \frac{A_0}{2}\int_{-\infty}^{+\infty} 0(x)\, dx = 0[\xi]$$

$$A_0 \int_{-\infty}^{+\infty} \underline{\overset{\cos}{\cancel{\sin}}(2\pi\xi_0 x)} \; \underline{\overset{\sin}{\cancel{\cos}}(2\pi\xi x)}\, dx = 0[\xi]$$

14

$\cos(2\pi\xi_0 x) \perp \sin(2\pi\xi x)$ FOR ALL ξ

ODD INPUT EVEN REFERENCE

$$A_0 \int_{-\infty}^{+\infty} \sin(2\pi\xi_0 x) \cdot \cos(2\pi\xi x)\, dx = 0[\xi]$$

ODD INPUT ODD REFERENCE

$$A_0 \int_{-\infty}^{+\infty} \sin(2\pi\xi_0 x) \cdot \sin(2\pi\xi x)\, dx$$

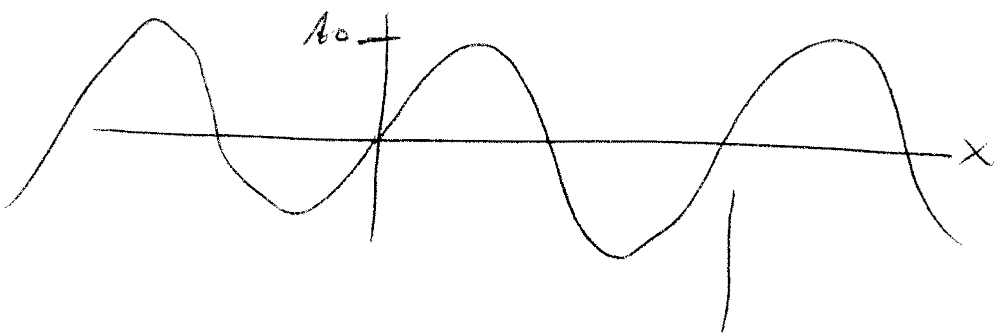
$$\sin(A)\sin(B) = \frac{1}{2}\overset{\cos}{\cancel{\sin}}(A-B) - \frac{1}{2}\overset{\cos}{\cancel{\sin}}(A+B)$$

$$A_0 \int_{-\infty}^{+\infty} \sin(2\pi \xi_0 x)\, \sin(2\pi \xi x)\, dx = \frac{A_0}{2} \int \cos(2\pi(\xi_0 - \xi)x)\, dx$$

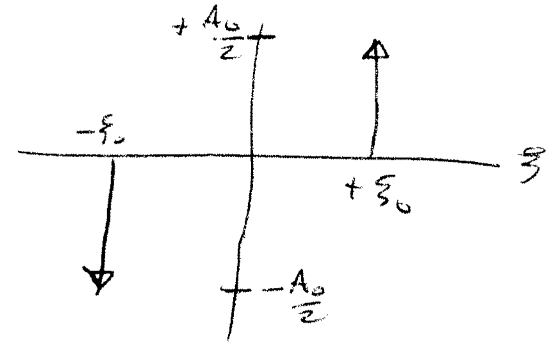
$$- \frac{A_0}{2} \int \cos(2\pi(\xi_0 + \xi)x)\, dx$$

$$= \frac{A_0}{2}\, \delta(\xi_0 - \xi) - \frac{A_0}{2}\, \delta(\xi_0 + \xi)$$

$$= \frac{A_0}{2}\, \delta(\xi - \xi_0) - \frac{A_0}{2}\, \delta(\xi + \xi_0)$$



$\xrightarrow{\mathcal{F}_1}$



INFINITE SUPPORT $\frac{1}{\xi_0}$

COMPACT SUPPORT

$$f(x) = A_0 \cos(2\pi\xi_0 x + \varphi_0)$$

$$= A_0 \cos(2\pi\xi_0 x)\cos\varphi_0 - A_0 \sin(2\pi\xi_0 x)\sin\varphi_0$$

$$= \underbrace{(A_0 \cos\varphi_0)\cos(2\pi\xi_0 x)}_{\text{even PART}} + \underbrace{(-A_0 \sin\varphi_0)\sin(2\pi\xi_0 x)}_{\text{ODD PART}}$$

$$\underbrace{\int f(x)\cos(2\pi\xi x)\,dx}_{\text{EVEN}} + \underbrace{\int f(x)\sin(2\pi\xi x)\,dx}_{\text{ODD}}$$

$$\frac{A_0}{2}\cos\varphi_0\left[\delta(\xi+\xi_0) + \delta(\xi-\xi_0)\right] + \left(-\frac{A_0}{2}\sin\varphi_0\right)\left[\delta(\xi-\xi_0) - \delta(\xi+\xi_0)\right]$$

$$\int f(x)\left[\underbrace{\cos(2\pi\xi x) + \sin(2\pi\xi x)}_{m(\xi,\, x)}\right]dx$$

$\longrightarrow$ HARTLEY $\quad CAS(2\pi\xi x)$

$$1\cdot\cos(2\pi\xi x) + 1\,\sin(2\pi\xi x)$$

$$1 \cdot \underbrace{\cos(2\pi\xi x)}_{\text{even ref.}} + i\, \underbrace{\sin(2\pi\xi x)}_{\text{ODD ref.}} = e^{+2\pi i \xi x}$$

$$\int_{-\infty}^{+\infty} f(x) \left(e^{+2\pi i \xi x}\right)^{*} dx$$ n.b. PROJECTS $f(x)$ ONTO $e^{+2\pi i \xi x}$

$$= \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{+\infty} f(x) \cos(2\pi\xi x)\, dx - i \int_{-\infty}^{+\infty} f(x) \sin(2\pi\xi x)\, dx$$

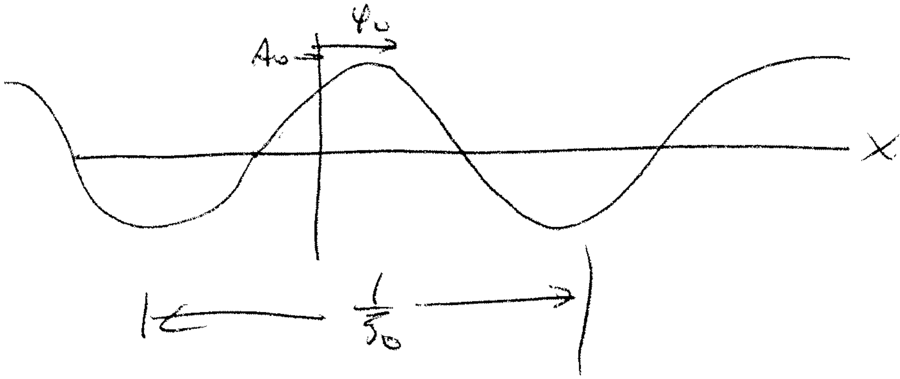
n.b.

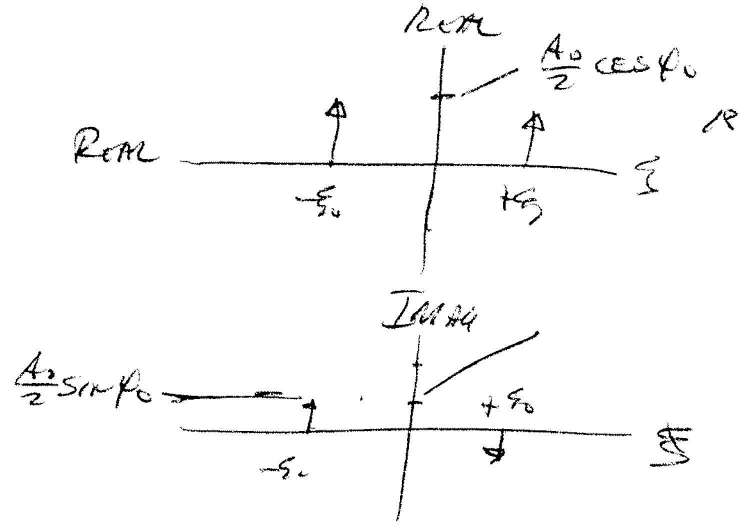
IF $f(x) = A_0 \cos(2\pi \xi_0 x + \varphi_0)$

$$\frac{A_0}{2}\cos\varphi_0 \left(\delta(\xi+\xi_0) + \delta(\xi-\xi_0) \right) - i\, \frac{A_0}{2}\sin\varphi_0 \left(\delta(\xi-\xi_0) - \delta(\xi+\xi_0) \right)$$

$$+\, i \left(\frac{A_0}{2}\sin\varphi_0\right)\left(\delta(\xi+\xi_0) - \delta(\xi-\xi_0) \right)$$

$$= \left(\frac{A_0}{2}\cos\varphi_0\right)\left(\delta(\xi+\xi_0) + \delta(\xi-\xi_0) \right)$$





$$\int_{-\infty}^{+\infty} \delta(x)\, e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{+\infty} \delta(x)\, e^{-2\pi i \xi \cdot 0}\, dx$$

$$= \int_{-\infty}^{+\infty} \delta(x) \cdot 1 \, dx = 1(\xi)$$

~~FOURIER ANALYSIS → FINDING AMPLITUDE OF~~ EACH SINUSOI
SINUSOIDAL COMPONENT
~~OF f(x)~~

FOURIER ANALYSIS – DETERMINE AMPLITUDE OF ALL SINUSOIDS IN f(x) AT ALL FREQUENCIES