SIMG-714 Information Theory for Imaging Science
Homework 3

1. Let $V(n,r)$ be the volume of an n-dimensional sphere of radius r.

   (a) Find the fraction of the volume of V that is in the shell at radius $[0.99r, r]$. as a function of n.

   (b) Compute the thickness of the outer shell that contains 99% of the volume as a function of n.

2. Let p be the probability of binary decision error in a binary symmetric channel, and let n be the length of a codeword. The number of binary patterns within a distance of $n(p+\varepsilon)$ of the original pattern is

$$N(np_\varepsilon) = \sum_{k=0}^{np_\varepsilon} \binom{n}{k} \leq 2^{nH(p_\varepsilon)}$$

   where

$$H(p_\varepsilon) = -p_\varepsilon \log p_\varepsilon - (1-p_\varepsilon)\log(1-p_\varepsilon)$$

   Discuss what this means in terms of the number of codewords of length n that can be used to transmit information with a specified error probability or better.

3. A linear block code is generated by a basis $\mathbf{G}$ whose elements $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the rows of the matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

   Every codeword is of the form $\mathbf{c} = m_1\mathbf{g}_1 + m_2\mathbf{g}_2 + m_3\mathbf{g}_3$ where $\{m_1, m_2, m_3\}$ are binary information digits and all arithmetic is done with binary modulo 2 arithmetic. Therefore, every vector $\mathbf{c} = \mathbf{mG}$ is a codeword and every codeword is of the form $\mathbf{c} = \mathbf{mG}$, where $\mathbf{m}$ is a row vector that contains the message digits.

   (a) Make a list of all of the vectors that are in the code.

   (b) Make a list of all of the vectors $\mathbf{r}$, that are closest to the code vector $\mathbf{c}_1 = [100101]$. Find the error vector $\mathbf{e}$ in each case that will cause $\mathbf{r} = \mathbf{c}_1 + \mathbf{e}$ and compute its probability in terms of the BSC bit error probability p.

   (c) Show that a an error vector $\mathbf{e}_k = \mathbf{r} - \mathbf{c}_k$ can be expressed in terms of its Hamming distance from $\mathbf{c}_k$, and that the error vector with the smallest Hamming distance given $\mathbf{r}$ is the most probable.

   (d) Develop a decoding table for $\mathbf{G}$.