SIMG-714 Information Theory for Imaging Science
Homework 3

1. Let $V(n,r)$ be the volume of an n-dimensional sphere of radius r.

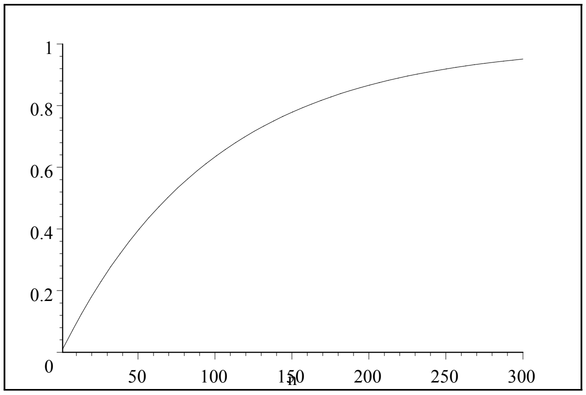
   (a) Find the fraction of the volume of V that is in the shell at radius $[0.99r, r]$. as a function of n.

   The volume of an n-dimensional sphere of radius r is proportional to r^n. Therefore, $V(n,r) = K(n)r^n$. The fraction of the volume in a concentric sphere of radius r_1 is

$$\frac{V(n,r_1)}{V(n,r)} = \frac{r_1^n}{r^n}$$

   Set $r_1 = 0.99r$ to find the volume fraction of the inner core. Subtract from 1 to get the volume fraction of the shell.

$$\frac{V_{\text{shell}}}{V} = 1 - \frac{(0.99r)^n}{r^n} = 1 - 0.99^n$$



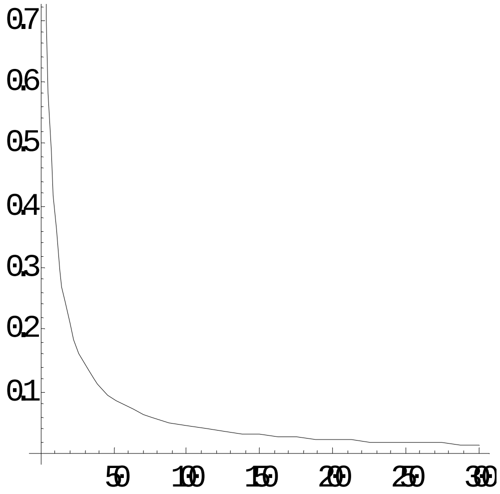
   (b) Compute the thickness of the outer shell that contains 99% of the volume as a function of n.

   The volume of the inner core must then be 0.01 and we have the ratio

$$\frac{V(n,r_1)}{V(n,r)} = \frac{r_1^n}{r^n} = 0.01$$

   Hence, $r_1 = (0.01)^{\frac{1}{n}} r$ and the thickness is $D_r = r - r_1 = r\left[1 - (0.01)^{\frac{1}{n}}\right]$.

A graph of D_r/r vs n is shown below.



2. Let p be the probability of binary decision error in a binary symmetric channel, and let n be the length of a codeword. The number of binary patterns within a distance of $n\,(p+\varepsilon)$ of the original pattern is

$$N\,(np_\varepsilon) = \sum_{k=0}^{np_\varepsilon} \binom{n}{k} \le 2^{nH(p_\varepsilon)}$$

where

$$H(p_\varepsilon) = -p_\varepsilon \log p_\varepsilon - (1-p_\varepsilon)\log(1-p_\varepsilon)$$

Discuss what this means in terms of the number of codewords of length n that can be used to transmit information with a specified error probability or better.

If there were no noise, then all of the 2^n binary words of length n would be available. Because of the channel noise, one has to provide a spacing between codewords of about $2n(p+\varepsilon)$, which is equivalent to placing each at the center of an n-dimensional sphere of radius $n\,(p+\varepsilon)$. The number of words used up in each sphere is given by the expression above. Hence, the fraction of the words available for information transmission is $2^n/2^{nH(p_\varepsilon)} = 2^{n(1-H(p_\varepsilon))}$. When the entropy of the channel noise approaches 1 then the rate of transmission must go to zero. However, the error rate can still be kept small. The number of bits per transmitted digit is $1 - H\,(p_\varepsilon)$, which is as close as you wish to the channel capacity. This word-counting analysis leads to an expression for the channel capacity.

3. A linear block code is generated by a basis $\mathbf{G}$ whose elements $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the rows of the matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Every codeword is of the form $\mathbf{c} = m_1\mathbf{g}_1 + m_2\mathbf{g}_2 + m_3\mathbf{g}_3$ where $\{m_1, m_2, m_3\}$ are binary information digits and all arithmetic is done with binary modulo 2 arithmetic. Therefore, every vector $\mathbf{c} = \mathbf{mG}$ is a codeword and every codeword is of the form $\mathbf{c} = \mathbf{mG}$, where $\mathbf{m}$ is a row vector that contains the message digits.

(a) Make a list of all of the vectors that are in the code.

There are eight possible message vectors, equal to the eight three-digit binary patterns. These are listed in the matrix M below. The codeword set is the array $\mathbf{C} = \mathbf{MG}$.

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) Make a list of all of the vectors $\mathbf{r}$, that are closest to the code vector $\mathbf{c}_1 = [100101]$. Find the error vector $\mathbf{e}$ in each case that will cause $\mathbf{r} = \mathbf{c}_1 + \mathbf{e}$ and compute its probability in terms of the BSC bit error probability p.

All of the words that are closest to codeword $\mathbf{c}_1$ are in the column below $\mathbf{c}_1$ in the standard array printed below. The error pattern leading to each is the row leader, next to the syndrome, to the left of each word. All of the error patterns have weight 1 except for one of weight 0 and one of weight 2. The probability, as a function of weight w, is $p(w) = p^w\,(1-p)^{6-w}$.

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000 000000 100101 010111 110010 001011 101110 011100 111001
100 000100 100001 010011 110110 001111 101010 011000 111101
010 000010 100111 010101 110000 001001 101100 011110 111011
110 101000 001101 111111 011010 100011 000110 110100 010001
001 000001 100100 010110 110011 001010 101111 011101 111000
101 100000 000101 110111 010010 101011 001110 111100 011001
011 001000 101101 011111 111010 000011 100110 010100 110001
111 010000 110101 000111 100010 011011 111110 001100 101001
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(c) Show that the probability of an error vector $\mathbf{e}_k = \mathbf{r} - \mathbf{c}_k$ can be expressed in terms of its Hamming distance from $\mathbf{c}_k$, and that the error vector with the smallest Hamming distance given $\mathbf{r}$ is the most probable.

If the digit errors are independent in the communication channel, then the probability of any specific error pattern with w errors is $p(w) = p^w (1-p)^{6-w}$. But w will be the Hamming weight of the error pattern and therefore the distance of r from c_k.

(d) Develop a decoding table for $\mathbf{G}$.

The decoding table is the syndrome column and first column of the standard array given above. Given a received word, r, compute the syndrome $s = rH^T$, where H is the parity check matrix. Use s to select the error pattern and add the error pattern to the received word to correct the error. The parity check matrix is

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$