SIMG-714 Information Theory for Imaging Science
Homework 2

1. An information source X has M symbols. Show that the source entropy is bounded by $H(X)$ □ $\log_2 M$.Explain the conditions under which equality holds.

2. Compute the average information transmission through a binary symmetric communication channel with source probabilities $P(x_1) = P(x_2) = 0.5$ and channel error probability $P(y_2|x_1) = P(y_1|x_2) = 0.9$. Explain your result.

3. Plot the amount of information that gets through a BSC with crossover probability $q = 0.05$ in terms of the probability $P(x_1) = p$. Explain the symmetry of your result.

4. A binary symmetric channel has erasure probability $q = 0.05$. Plot the amount of information that gets through the channel in terms of the probability $P(x_1) = p$.

5. A discrete channel is characterized by the matrix

$$\mathcal{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

   (a) Given the input probabilities $p(x_1) = 1/2$, $p(x_2) = p(x_3) = 1/4$, find $I(X;Y)$.
   (b) Construct the decision rules for an ideal observer and compute the error probability. An ideal observer chooses x_i to maximize $P(x_i|y_j)$ for each y_j.
   (c) Compute the channel capacity for this channel.

6. A number of identical binary symmetric channels are cascaded, so that the output of one channel is the input to the next. Let p_0 be the probability that $x = 0$ is the input to the first channel. Find the probability p_n that the input to channel n is 0. Find an explicit expression for p_n in terms of p_0 and the crossover probability, β, of a single BSC stage. Show that $p_n \to 1/2$ regardless of the value of p_0 if $\beta > 0$ when n becomes large.

7. A number of communication channels are cascaded so that the output of one channel is the input of the next. Let P_k be the channel matrix of channel k. Show how to compute the channel matrix of the cascaded combination in terms of the individual channel matrices. Apply your result to the determination of the channel matrix of the cascaded BSC channels in the previous problem. Does your result agree with the conclusions of problem 6?