SIMG-713 SPRING 2002

1 REVIEW TOPICS FOR MIDTERM EXAM

1. Concept of an experiment that produces outcomes that can be different on each trial.
2. Concept of events as sets of possible experiment outcomes.
3. Probability as a number associated with events.
   (a) Properties of probability
   (b) Joint probabilities
   (c) Conditional probabilities
   (d) Statistical independence
   (e) Mutually exclusive events
4. Random variables
5. Averages over random variables.
   (a) Computation of averages and moments
   (b) Physical interpretation of mean and variance
   (c) Correlation and covariance
6. Averages over functions of a random variable
   (a) Moments
   (b) Mean, variance, standard deviation
   (c) $E[Y = h(X)]$ directly from the pdf $f(X)$.
   (d) Characteristic function
7. Modeling of discrete events and random arrivals
   (a) Binomial distribution
   (b) Poisson distribution
   (c) Normal approximation to binomial
8. Law of large numbers

2 Review Questions

1. What is the meaning of *outcome*, *trial, event, sample space* in probability modeling?

2. Let A and B be events. Determine whether the following statements can be true. If false, give a reason.

   (a) $P(A \cap B) = P(A)$

   (b) $P(A \cup B) = P(A) + P(B)$

   (c) $P(A \cup B) = P(A)$

   (d) $P(A|B) = P(A)$

   (e) If $P(A|B) = P(A)$ then $P(A \cup B) = P(A) + P(B)$

   (f) If $P(A \cup B) = P(A) + P(B)$ then $P(A \cap B) = 0$

   (g) $P[(A \cup B)^c] = P(A^c)P(B^c)$

   (h) $P[(A \cap B)^c] = P(A^c) + P(B^c)$

3. Let X be a random variable associated with an experiment. Assume that a probability is assigned to every possible outcome. How can we use that information to compute the probability that X falls in a prescribed interval on the number line? Does a statement like $X \leq x$ correspond to a set of outcomes? *i.e.* is it an event?

4. What is the probability density function that is associated with the distribution function $Q(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$? How can this function be used to find $P(a < X < b)$ for experiments where that density function holds.

5. Let X be a normal random variable with mean 5 and standard deviation 10. Find the probability that $2 < X < 6$.

6. Let X be an exponential random variable with probability density

   $$f(x) = 2\,e^{-2x} step(x)$$

   Fine the expected value of the random variable $Y = \sin 2\pi X$.

7. Let X be an exponential random variable with probability density

   $$f(x) = 2\,e^{-2x} step(x)$$

   What is the probability density function of $Y = e^X$?

8. Use the method of the characteristic function to find the moments of a rectangular probability distribution. Verify your answer with direct calculation.

9. Given a probability density function $f(x, y)$ for random variables X and Y, how do you find the probability density function $f_X(x)$? How do you find conditional density functions such as $f(y|x)$?

10. Construct a problem for your friend to solve concerning the law of large numbers.

11. What are the three assumptions concerning the Poisson distribution?

12. A certain detector has a logarithmic detector characteristic, $h(x) = \ln(x + 1)$, $x \geq 0$. Write suitable equations to estimate the mean response when the average rate of photon arrival is q. Continue the analysis with a description of how you would compute the comparative noise level for this system.

3 Questions from Old Exams

1. A random process $X(n)$ is generated by repeatedly drawing numbers from a source that provides values that are uniformly distributed over $[0, 1]$. Each number is statistically independent of all others in the sequence. A new random sequence is generated by $Y(n) = (2X(n) - 1)^2$. Find the average value of the sequence $Y(n)$.

2. A detector has two sensor cells whose outputs are connected to form the sum and difference of the sensor responses. If the sensor responses are X and Y then the outputs are $S = X + Y$ and $D = X - Y$. Assume that X and Y are described by independent Poisson distributions with parameters q_x and q_y, respectively.

   (a) Find the mean of S and D.

   (b) Find the variance of S and D.

   (c) Write an expression for the probability distribution of D and simplify. Do not compute a closed form for any summation that may be in the expression.

   (d) Determine whether or not the event $S = a$ is statistically independent of the event $D = b$.

3. A random variable X has the discrete probability distribution

$$P[X = k] = Ae^{-kb} \quad \text{for } k \geq 0$$
$$P[X = k] = 0 \quad \text{for } k < 0$$

   (a) Find an expression for the value of A. You may want to make use of the sum $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ if $|r| < 1$.

   (b) Find the characteristic function $M_X(ju)$ of this probability distribution.

(c) Find the mean value of the distribution by a direct sum using the probabilities and by using the characteristic function. (You should get the same answer!)

4. For a certain random variable X,

$$P[X \leq x] = \frac{e^x}{1 + e^x}$$

Set up the expressions that need to be evaluated to compute the variance of X. Do not carry out any integrals or summations.

5. A random variable X has a normal probability density function with zero mean and unit variance. The random variable $Y = 2X + 3$ is formed.

   (a) Find the probability density function of Y

   (b) Write an expression that can be used to compute the probability $P[5 < Y < 9]$ in terms of the error function[1]

6. Radioactive decay events associated with a certain material were observed. It was noted that 1000 events occurred in 25 seconds.

   (a) Let X be a random variable equal to the number of events in τ seconds. Construct an appropriate probability model for X and support your analysis.

   (b) Compute the probability of zero events in 50 milliseconds.

7. A random variable X has the probability density function (where $a > 0$)

$$f_X(x) = Be^{-a|x|}$$

   (a) Determine the value of B in terms of the parameter a.

   (b) Find the characteristic function. The correct answer is one of the choices below.

$\frac{a}{\sqrt{a^2+u^2}}$	$\frac{a}{a^2+u^2}$	$\frac{a^2}{a^2+u^2}$	$\frac{2iu}{a^2+u^2}$
(a)	(b)	(c)	(d)

   (c) Find the variance of X.

8. A certain experiment A has two possible outcomes which we will call H and T, with $P(H) = 0.02$. Consider another experiment, which we will call B, that consists of 7200 repetitions of A. Let X be the number of times H occurs in B.

   (a) Construct an appropriate probability model for X and support your analysis.

[1] $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du$

(b) Assume that the probability $P[X = k]$ can be approximated by a normal probability function. Construct an appropriate expression for this normal approximation.

9. Random variables X and Y have the joint probability distribution function (where both a and b are positive)

$$F_{XY}(x, y) = \begin{cases} 1 - e^{-ax} - e^{-by} + e^{-ax-by} & \text{for} \quad x \geq 0,\ y \geq 0 \\ 0 & \qquad\quad \text{elsewhere} \end{cases}$$

(a) Find the joint probability density function $f_{XY}(x, y)$.

(b) Find the probability density function $f_X(x)$.

(c) Determine whether the random variables X and Y are statistically independent.

10. A random variable X has an exponential probability density function ($a > 0$)

$$f_X(x) = ae^{-ax}\text{step}(x)$$

Find the probability density function of the random variable $Y = \sqrt{X}$