SIMG-713
Homework 6
Spring 2002

1. A random telegraph signal $x(t)$ that takes on the values $\pm A$ has the autocorrelation function $R_{xx}(\tau) = A^2 e^{-\lambda|\tau|}$

   (a) What is the autocorrelation function of $y(t) = Bx(t) + C$ where B and C are constants?

   (b) How can you identify the mean value of a random process by inspection of it autocorrelation function?

   (c) How do the covariance functons of $x(t)$ and $y(t)$ differ?

2. A sample function from a random process can appear quit ordinary. Consider the random process $X(t,\theta)$ in which each sample function is a sinusoid $x(t) = A\sin(\omega t + \theta)$ where A and ω are constants and θ is a random variable that is uniformly distributed over $[-\pi, \pi]$.

   (a) Compute $E[X(t,\theta)]$ by

   $$E[X(t,\theta)] = \int_{-\pi}^{\pi} A\sin(\omega t + \theta)\, f_\theta(\theta)\, d\theta$$

   Does it make sense that the result does not depend upon ω or t? Explain.

   (b) Compute the autocorrelation function $R_{xx}(t_1, t_2)$. Determine whether the process is wide-sense stationary.

   (c) Compute the mean-squared value of the random process.

3. Let $X(t)$ and $Y(t)$ be two wide-sense stationary random processes. Give at least two reasons that the following matrix cannot be the covariance matrix.

   $$C = \begin{bmatrix} A^2 \cos\tau & 2A^2 \cos\frac{3\tau}{2} \\ 2A^2 \cos\frac{3\tau}{2} & A^2 \sin\tau \end{bmatrix}$$

   For reference, the covariance matrix is

   $$\Lambda = \begin{bmatrix} \mathrm{var}(X) & \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,Y) & \mathrm{var}(Y) \end{bmatrix}$$

4. Photons arrive at a detector at a rate λ photons/second. Each photon causes the detector to emit an electronic pulse. Assume that the pulses are triangular in shape with height A and width $w << 1/\lambda$. Describe the random process

   $$Y(t_0, T) = \frac{1}{T}\int_{t_0}^{t_0+T} X(t)\, dt$$

   In particular, how do you expect the mean and variance of Y to behave as T becomes arbitrarily large?

5. A random process $Y(t)$ is produced by passing white noise $X(t)$ with autocorrelation function $R_{xx}(\tau) = \sigma_x^2 \delta(\tau)$ through a linear shift-invariant filter that has impulse response $h(t)$.

   (a) Show that the cross correlation function $R_{xy}(\tau) = E[X(t)Y(t+\tau)] = \sigma_x^2 h(\tau)$. Note that this provides an interesting possibility for the measurement of an impulse response.

(b) Find the autocorrelation function $R_{yy}(\tau)$ in terms of the impulse response.

(c) Find $R_{yy}(\tau)$ when the impulse response is $h(t) = Ae^{-ct}\,\mathrm{step}(t)$.

6. A digital filter is described by the difference equation

$$y(n) - 2r\cos(\omega)\,y(n-1) + r^2 y(n-2) = x(n) - r\cos(\omega)\,x(n-1)$$

(a) Draw the digital filter block diagram.

(b) Find the system function $H(z)$.

(c) Assume that $y(-2) = y(-1) = 0$. Compute and plot the $y(n)$ for $x(n) = [1, 0, 0, 0, \cdots]$. Use $\omega = \pi/10$ and $r = 0.9$. Plot enough points so that you can see the behavior of the response function.