SIMG-713
Homework 4
Solutions

1. Let S be the number of successes in 100 Bernoulli trials with $p = 0.4$ equal to the probability of success on any one trial. Find the probability that the number of successes exceeds 45.

   Solution: (Method #1) Approximate the binomial distribution by a normal distribution with the same mean and standard deviation. Use tables of the normal distribution or calculation with the error function to find the answer. To use the normal distribution, we first compute the value of the point corresponding to 45.5 (half-way between 45 and 46) on a standard normal distribution. That point is

   $$z = \frac{45.5 - \mu}{\sigma} = \frac{45.5 - np}{\sqrt{np\left(1-p\right)}} = \frac{45.5 - 40}{\sqrt{24}} = 1.1227$$

   The probability that the number of successes exceeds 45 is

   $$1 - Q\left(z\right) = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(z/\sqrt{2}\right) = 0.13078$$

   (Method #2) Another approach to solving the problem is to sum up binomial probabilities. The difficulty in doing that is computing the large numbers that appear in the expressions. This can be relieved by using a recursion formula.

   $$\begin{aligned} b(n,k,p) &= \frac{n!}{k!\left(n-k\right)!}p^k q^{n-k} \\ &= \frac{\left(n-k+1\right)p}{kq} b(n,k-1,p) \end{aligned}$$

   with $q = 1 - p$. We know that $b\left(n,0,p\right) = q^n$ and $b\left(n,n,p\right) = p^n$. We can write a computer program that can compute any value by recursion from one end or the other. One can avoid multiplications and some of the large number problems by computing with logarithms

   $$\log b(n,k,p) = \log(n-k+1) - \log k + \log p - \log q + \log b(n,k-1,p)$$

   An IDL program to compute the binomial probabilities using this technique is listed below. With this technique we find that

   $$\sum_{k=46}^{100} b(100,k,0.4) = 0.1311$$

```
FUNCTION bernoulli,n,p

;Compute the binomial distribution for Bernoulli trials.

;Initialize an array to hold the results
```

```
b=dblarr(n+1)
;Find log(p)
lp=alog(p)
;Compute the first array element
b[0]=n*lq
;Initialize the binomial coefficient
r=double(1.0)
;Compute the logarithms of all of the probabilities
FOR k=1,n DO BEGIN
 r=r*(n-k+1)/k
 lr=alog(r)
 b[k]=lr+k*lp+(n-k)*lq
END
;Invert the log operations and return the result
RETURN,exp(b)
END
```

2. Consider a sequence of Bernoulli trials with $p = 0.5$. Determine the number of trials that you must conduct such the probability $P[S_n/n > 0.51] \leq 0.01$.

   Solution: The condition $S_n/n > 0.51$ is equivalent to $\frac{S_n - 0.5n}{\sigma} > \frac{0.01n}{\sigma}$. The term on the left is a normalized random variable with zero mean and unit variance. For large n the probability distribution can be approximated by a normal distribution. From a table of the distribution function for a normal random variable we find that $P[Z > 2.36] \approx 0.01$. Hence, using $\sigma = \sqrt{np(1-p)} = \sqrt{n}/2$, we find

   $$2.36 > \frac{0.01n}{\sigma} = 0.02\sqrt{n}$$

   from which we can solve for n. We must have $n \geq (50 * 2.36)^2 = 13,924$ to satisfy the inequality.

3. A sensor array with 1000 elements is struck by 1000 photons. Estimate the probability that a given element is struck by at least three photons.

   Solution: This is a Poisson process with $\lambda = 1$ photon/element. The probability that a given element is struck by k photons is

   $$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

   The probability of 3 or more can be found by computing the probability of 0 or 1 or 2 and subtracting from unity. Since these events are mutually exclusive,

   $$P(k=0) + P(k=1) + P(k=2) = (1 + 1 + \frac{1}{2})e^{-1} = 0.92$$

   Hence, $P(k \geq 3) = 1 - 0.92 = 0.08$

4. A book contains 500 pages and contains on the average 1 misprint every 10 pages. Estimate the probability that a random selection of 5 pages will contain 2 misprints. Does it matter whether the five pages are contiguous?

   Solution: This can be modeled as a binomial distribution with a probability of success $S = 0.1$. The probability of two successes in five trials is $\binom{5}{2}(0.1)^2(0.9)^3 = 0.072\,9$

5. Raindrops strike an area of 10 square centimeters at a rate of 1 drop per second. What is the probability that a given area of 1 square centimeter will not be struck during a time interval of length 1 minute?

   Solution: This can be modelled as a Poisson process with $\lambda = 0.1$ drop/cm^2/sec. Then

$$P(k) = \frac{(0.1)^k e^{-0.1T}}{k!}$$

   In a time of $T = 60$ seconds, $P(k = 0) = e^{-(0.1)(60)} = 2.\,48 \times 10^{-3}$.

6. An array of 100 detectors is placed in a photon stream. The average number of photons arriving at a detector is $q = 5$. What is the standard deviation in the number of photons per detector?

   Solution: The number of detectors is not a factor in this problem since the answer is for an individual detector. For a Poisson distribution the variance is equal to the mean. Hence, $\sigma^2 = q = 5$ and $\sigma = \sqrt{5} = 2.\,24$.

7. A certain photomultiplier emits N electrons when it is struck by one photon. The number N is a random variable with the probability distribution given by the table below.

N	0	1	2	3
$P(N)$	.1	.2	.3	.4

   The detector is actually struck by X photons per second, with $E[X] = 4$, which leads to the production of Y electrons. What is $E[Y]$, the expected number of electron emissions per second?

   Solution: $E[Y]$ can be calculated in two ways:

$$E[Y] = \sum_{y \in S_y} yP[Y = y] = \sum_{x \in S_x} h(x)\, P[X = x]$$

   The second form is easiest if we can determine the transfer function, $h(x)$. Unfortunately, this is difficult for this problem. Therefore, we proceed by finding $P(Y)$. This can be done by

$$P[Y = y] = \sum_{x \in S_x} P[x = X]\, P[y|x]$$

   We know that X has a Poisson distribution. We will assume that the detector produces $y = 0$ electrons when there are $x = 0$ photons. Thus, $P[y|0] = \delta(y)$. When $x = 1$ the probability distribution on Y is given by $P(N)$ above.

Y	0	1	2	3
$P(Y\|1)$	.1	.2	.3	.4

We will assume that two or more photons act independently. For $x \geq 2$ we have $Y = Y_1 + Y_2 + \cdots + Y_x$ where Y_i is the number of electrons generated by the i^{th} photon acting independently. Each of the terms in the sum has the probabilities in the table above. The probability distribution for the sum, which yields $P(y|x)$, is found by convolving the $P(Y|1) = P(N)$ distribution with itself x times. Once $P(Y = y)$ has been computed, one can easily compute $E[Y]$ by the formula above. An IDL program that computes the various probabilities is listed below.

```
FUNCTION PROB7,q,pp,py,pe
;Set the number of values to compute for p[X=k].  The number needed
;depends on q.  A reasonable number is about 3 sigma above the mean.
nv=ceil(q+3*sqrt(q))
;Set the electron distribution per photon.  This is the essence of the
;model for the detector.
pn=[0.1,0.2,0.3,0.4]
;Construct an array that will hold the p(y|x) values, where x is the
;row index and y is the column index.  The maximum number of y values
;is 3*nv+1.
py=fltarr(3*nv+1,nv+1)
;When x=0 the only possibility is y=0.  Hence,p(0|0)=1.
py[0,0]=1
;When x=1, p(y|1) corresponds to the vector pe.
py[0:3,1]=pn
;Fill in the rest of the rows by convolving.
FOR k=2,nv DO $
 py[0:3*k,k]=convolve(pn,py[0:3*(k-1),k-1])
;Compute the distribution on photon arrivals assuming
;average rate q.  All probabilities p(x) represented by
;the vector pp can be computed at once.
k=findgen(nv+1)
pp=q^k*exp(-q)/factorial(k)
;Multiply p(x)p(y|x) and sum over x to get p(y) for each
;value of y.  This is easily done by array multiplication
pe=pp##py
;Compute the mean value by multiplying y*p(y) and summing.
;Again, easily done with array multiplication.
av=findgen(3*nv+1)##transpose(pe)
return,av
```

```
END
```

The input photon distribution is given by the probability vector `pp`. This is listed in the table below.

X	0	1	2	3	4	5
$P(X)$	0.0183	0.0733	0.1465	0.1954	0.1954	0.1563
X	6	7	8	9	10	
$P(X)$	0.1042	0.0595	0.0298	0.0132	0.0053	

The array $P(Y|X)$ is given by the array `py`. The result is listed below. The transpose of the array is printed, so that the values for X run across the columns and the values for Y run down the columns. This is for easier printing.

$y\|x$	0.0000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000	10.0000
0.0000	1.0000	0.1000	0.0100	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.2000	0.0400	0.0060	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
2.0000	0.0000	0.3000	0.1000	0.0210	0.0036	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
3.0000	0.0000	0.4000	0.2000	0.0560	0.0120	0.0022	0.0004	0.0001	0.0000	0.0000	0.0000
4.0000	0.0000	0.0000	0.2500	0.1110	0.0310	0.0069	0.0013	0.0002	0.0000	0.0000	0.0000
5.0000	0.0000	0.0000	0.2400	0.1740	0.0648	0.0177	0.0040	0.0008	0.0001	0.0000	0.0000
6.0000	0.0000	0.0000	0.1600	0.2190	0.1124	0.0383	0.0103	0.0024	0.0005	0.0001	0.0000
7.0000	0.0000	0.0000	0.0000	0.2040	0.1608	0.0704	0.0228	0.0061	0.0014	0.0003	0.0001
8.0000	0.0000	0.0000	0.0000	0.1440	0.1905	0.1109	0.0437	0.0136	0.0036	0.0009	0.0002
9.0000	0.0000	0.0000	0.0000	0.0640	0.1840	0.1497	0.0736	0.0271	0.0082	0.0022	0.0005
10.0000	0.0000	0.0000	0.0000	0.0000	0.1376	0.1720	0.1086	0.0478	0.0167	0.0050	0.0013
11.0000	0.0000	0.0000	0.0000	0.0000	0.0768	0.1666	0.1403	0.0753	0.0307	0.0103	0.0030
12.0000	0.0000	0.0000	0.0000	0.0000	0.0256	0.1328	0.1581	0.1059	0.0508	0.0195	0.0064
13.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0832	0.1537	0.1325	0.0761	0.0337	0.0124
14.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0384	0.1270	0.1470	0.1031	0.0531	0.0220
15.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0102	0.0868	0.1434	0.1258	0.0764	0.0362
16.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0468	0.1216	0.1379	0.1003	0.0547
17.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0184	0.0880	0.1349	0.1201	0.0762
18.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0041	0.0529	0.1167	0.1304	0.0977
19.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0251	0.0881	0.1278	0.1150
20.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0086	0.0569	0.1123	0.1239
21.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0016	0.0306	0.0876	0.1217
22.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0129	0.0597	0.1083
23.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0039	0.0349	0.0866
24.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0007	0.0170	0.0616
25.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0065	0.0384
26.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0018	0.0205
27.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0091
28.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0032
29.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0008
30.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

The values for $P(x, y)$ are formed by multiplying down each column by $P(x)$. The values for $P(Y)$ are then formed by summing each row over all values of X. The values are given by the array `pe` in the program. These values are printed in the table below.

0	1	2	3	4	5	6	7	8	9
0.0273	0.0219	0.0415	0.0723	0.0656	0.0851	0.0954	0.0851	0.0882	0.0814
10	11	12	13	14	15	16	17	18	19
0.0685	0.0612	0.0503	0.0397	0.0319	0.0241	0.0178	0.0132	0.0093	0.0064
20	21	22	23	24	25	26	27	28	29
0.0043	0.0028	0.0017	0.0010	0.0006	0.0003	0.0001	0.0001	0.0000	0.0000
30									
0.0000									

The mean value can be computed from the above table by $E\left[Y\right] = \sum_{k=0}^{30} kP[Y = k] = 7.94$.

8. What is the variance of Y for the above problem at $E\left[X\right] = 4$?

   Solution: The variance can be computed from the table above by $E[Y^2] = \sum_{k=0}^{30} k^2 P[Y = k] = 82.47$. The variance is then $var\left[Y\right] = E\left[Y^2\right] - E^2\left[Y\right] = 82.47 - (7.94)^2 = 19.5$

9. Suppose that the brightness of the input photon stream is changed to $E\left[X\right] = 5$. Find the new value of $E\left[Y\right]$ and use it to estimate "gain" g of the system. That is, what is the rate of change of $E\left[Y\right]$ with a change in $E\left[X\right]$?

   Solution: The mean value is computed in the same manner as Problem 7. The result is $E\left[Y\right] = 9.95$. The gain is $g = 9.95 - 7.94 = 2.01$ electrons/photon at this operating point.

10. Compute the ratio

$$r = \frac{g^2 \mathrm{var}\left[X\right]}{\mathrm{var}\left[Y\right]}$$

using the results of problems 8 and 9. Would you expect this ratio to be larger or smaller than 1.0. Explain.

For a Poisson process, $\mathrm{var}[X] = E\left[X\right]$. We can calculate r at both input levels $E\left[X\right] = 4$ and $E\left[X\right] = 5$ and compare:

$$DQE = \left.\frac{g^2 \texttt{var}[X]}{\texttt{var}[Y]}\right|_{E[X]=4} = \frac{(2.01)^2 \times 4}{19.5} = 0.83$$

$$DQE = \left.\frac{g^2 \texttt{var}[X]}{\texttt{var}[Y]}\right|_{E[X]=5} = \frac{(2.01)^2 \times 5}{24.58} = 0.82$$

It is somewhat surprising that this detector, with its noisy emission process, has a DQE ratio that is this large.