SIMG-713

Homework 3

1. Let U be a discrete random variable and let $V = aU + b$ where a and b are constants. Show that $E[V] = aE[U] + b$.

2. Let U be a discrete random variable. Show that $|E[U]| \leq E[|U|]$ and specify the conditions that must be true for equality to hold. [Hint: Use the triangle inequality $|\sum_k r_k| \leq \sum_k |r_k|$].

3. Let $V = a_1U_1 + a_2U_2 + \cdots + a_nU_n$. Carry out an analysis similar to that of Example 3.2.2 to find $E[V]$ in terms of the expectations of the U_k.

4. Suppose that U_k is a binomial random variable that takes on the value 1 with probability p and the value 0 with probability $(1-p)$. Let $V = U_1 + U_2 + \cdots + U_n$ be the sum of n such binomial random variables. Show that $E[V] = np$.

5. Let U_1 and U_2 be statistically independent random variables, and let $V = U_1U_2$. Show that $E[V] = E[U_1]E[U_2]$. Make specific use of the assumption of statistical independence.

6. Let X be a discrete random variable with the Poisson probability distribution

$$P[X = k] = \frac{\mu^k e^{-\mu}}{k!} \quad k = 0, 1, 2, \ldots$$

   (a) Show that $\sum_{k=0}^{\infty} P[X = k] = 1$

   (b) Show that $E[X] = \mu$

   (c) Show that $\mathrm{var}(U) = \mu$. That is, the expected value and the variance of a Poisson distribution are equal.

7. Let X be a normal random variable with the probability density function

$$f_X(x) = \frac{1}{s\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2s^2}\right]$$

   (a) Show that $\int_{-\infty}^{\infty} f_X(x)dx = 1$. Hint: Consider a function $f_{XY}(x, y) = f_X(x) f_Y(y)$ with X and Y identically distributed and statistically independent. It turns out to be easier to do the integral $\int\int f_{XY}(x, y)\, dxdy$ than it is to do $\int f_X(x)\, dx$ because you can make a change of variable that converts the 2D integral from rectangular to polar coordinates. The integration in polar coordinates is very easy. If you can do this you will find $\int\int f_{XY}(x, y)\, dxdy = 1$, from which you reach the desired conclusion.

   (b) Show that $E[X] = a$, so that writing μ in the position occupied by a is a sensible thing to do.

(c) Show that $\text{var}(X) = s^2$ so that writing σ in the position occupied by s is a sensible thing to do.

8. Verify the results for the mean and standard deviation of the Rayleigh distribution that are given in Example 3.4.1. Plot the Rayleigh distribution for $b = 1, 4, 9, 25$ and observe the changes in the graph.

9. Calculate the characteristic function $M_X(j\omega) = E\left[e^{j\omega X}\right]$ for a random variable X that has a Poisson distribution

$$P[X = k] = \frac{\mu^k e^{-\mu}}{k!} \quad k = 0, 1, 2, \ldots$$

Use the characteristic function to compute the moments $E[X]$, $E[X^2]$ and $E\left[X^3\right]$.