SIMG-713
Homework 3
Solutions

1. Let U be a discrete random variable and let $V = aU + b$ where a and b are constants. Show that $E[V] = aE[U] + b$. Expectation is a linear operation. For example, suppose that U is a continuous random variable. Then

$$\begin{aligned} E[V] &= \int_{-\infty}^{\infty} f_U(u)(au+b)du \\ &= a\int_{-\infty}^{\infty} f_U(u)udu + b\int_{-\infty}^{\infty} f_U(u)du \\ &= aE[U] + b \end{aligned}$$

2. Let U be a discrete random variable. Show that $|E[U]| \leq E[|U|]$ and specify the conditions that must be true for equality to hold. [Hint: Use the triangle inequality $|\sum_k r_k| \leq \sum_k |r_k|$].

$$\begin{aligned} |E[U]| &= |\textstyle\sum_k u_k f_U(u_k)| \\ &\leq \textstyle\sum_k |u_k f_U(u_k)| \quad \text{Equality if all positive terms} \\ &= \textstyle\sum_k |u_k| f_U(u_k) \quad \text{Since } f_U(u_k) \text{ is positive} \end{aligned}$$

Equality will hold if $f_U(u) = 0$ for $u < 0$.

3. Let $V = a_1U_1 + a_2U_2 + \cdots + a_nU_n$. Carry out an analysis similar to that of Example 3.2.2 to find $E[V]$ in terms of the expectations of the U_k. Let $f_{\mathbf{U}}(U_1, U_2, \ldots, U_n)$ be the joint probability density function for $\{U_1, U_2, \ldots, U_n\}$. Then

$$\begin{aligned} E[V] &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} a_k u_k\right) f_{\mathbf{U}}(u_1, u_2, \ldots, u_n)du_1du_2\cdots du_n \\ &= \sum_{k=1}^{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} a_k u_k f_{\mathbf{U}}(u_1, u_2, \ldots, u_n)du_1du_2\cdots du_n \\ &= \sum_{k=1}^{n} a_k \int_{-\infty}^{\infty} u_k f_{U_k}(u_k)du_k \\ &= \sum_{k=1}^{n} a_k E[U_k] \end{aligned}$$

4. Suppose that U_k is a binomial random variable that takes on the value 1 with probability p and the value 0 with probability $(1-p)$. Let $V = U_1 + U_2 + \cdots + U_n$ be the sum of n such binomial random variables. Show that $E[V] = np$. The expected value of U_k is p

$$E[U_k] = p \cdot 1 + (1-p) \cdot 0 = p$$

for every k, since all of the random variables have the same distribution. Then

$$E[V] = \sum_{k=1}^{n} E[U_k] = \sum_{k=1}^{n} p = np$$

5. Let U_1 and U_2 be statistically independent random variables, and let $V = U_1U_2$. Show that $E[V] = E[U_1]E[U_2]$. Make specific use of the assumption of statistical independence.

$$\begin{aligned}
E[V] &= \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{U_1U_2}(u_1, u_2) u_1 u_2 du_1 du_2 \\
&= \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{U_1}(u_1) f_{U_2}(u_2) u_1 u_2 du_1 du_2 \quad \text{Use S.I. assumption} \\
&= \textstyle\int_{-\infty}^{\infty} f_{U_1}(u_1) u_1 du_1 \int_{-\infty}^{\infty} f_{U_2}(u_2) u_2 du_2 \\
&= E[U_1]E[U_2]
\end{aligned}$$

6. Let X be a discrete random variable with the Poisson probability distribution

$$P[X = k] = \frac{\mu^k e^{-\mu}}{k!} \qquad k = 0, 1, 2, \ldots$$

(a) Show that $\sum_{k=0}^{\infty} P[X = k] = 1$

$$\begin{aligned}
\textstyle\sum_{k=0}^{\infty} \frac{\mu^k e^{-\mu}}{k!} &= e^{-\mu} \textstyle\sum_{k=0}^{\infty} \frac{\mu^k}{k!} \\
&= e^{-\mu} e^{\mu} = 1 \qquad \text{Use series for } e^{\mu}
\end{aligned}$$

(b) Show that $E[X] = \mu$

$$\begin{aligned}
E[X] &= \textstyle\sum_{k=0}^{\infty} kP[X = k] \\
&= e^{-\mu} \textstyle\sum_{k=0}^{\infty} k \frac{\mu^k}{k!} \\
&= \mu e^{-\mu} \textstyle\sum_{k=1}^{\infty} \frac{\mu^{k-1}}{(k-1)!} \quad \text{cancel } k\text{; factor out } \mu \\
&= \mu e^{-\mu} \textstyle\sum_{m=0}^{\infty} \frac{\mu^m}{(k)!} \qquad \text{Let } m = k - 1 \\
&= \mu e^{-\mu} e^{\mu} = \mu \qquad \text{Use series for } e^{\mu}
\end{aligned}$$

(c) Show that $\mathrm{var}(U) = \mu$. That is, the expected value and the variance of a Poisson distribution are equal.

$$\begin{aligned}
E[X^2] &= \textstyle\sum_{k=0}^{\infty} k^2 P[X=k] & \\
&= e^{-\mu} \textstyle\sum_{k=0}^{\infty} k^2 \frac{\mu^k}{k!} & \\
&= e^{-\mu} \textstyle\sum_{k=1}^{\infty} k \frac{\mu^k}{(k-1)!} & \text{cancel one } k \\
&= e^{-\mu} \textstyle\sum_{k=1}^{\infty} \frac{(k-1)+1}{(k-1)!} \mu^k & \text{rearrange} \\
&= \mu^2 e^{-\mu} \textstyle\sum_{k=2}^{\infty} \frac{1}{(k-2)!} \mu^{k-2} + \mu e^{-\mu} \textstyle\sum_{k=1}^{\infty} \frac{1}{(k-1)!} \mu^{k-1} & \text{algebra} \\
&= \mu^2 e^{-\mu} \left(\textstyle\sum_{k=0}^{\infty} \frac{1}{k!} \mu^k \right) + \mu e^{-\mu} \left(\textstyle\sum_{k=0}^{\infty} \frac{1}{k!} \mu^k \right) & \text{Index change} \\
&= \mu^2 + \mu &
\end{aligned}$$

Now calculate $\mathrm{var}[X] = E[X^2] - E^2[X] = (\mu^2 + \mu) - \mu^2 = \mu$.

7. Let X be a normal random variable with the probability density function

$$f_X(x) = \frac{1}{s\sqrt{2\pi}} \exp\left[-\frac{(x-a)^2}{2s^2} \right]$$

(a) Show that $\int_{-\infty}^{\infty} f_X(x)dx = 1$ Set up the integral and simplify by using the change of variable $u = (x-a)/s\sqrt{2}$.

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)dx &= \frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2s^2}} dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du
\end{aligned}$$

This integral cannot be done in closed form. However, the square of the integral actually can be evaluated by making a change to polar coordinates.

$$\begin{aligned}
\left(\tfrac{1}{\sqrt{\pi}} \textstyle\int_{-\infty}^{\infty} e^{-u^2} du \right)^2 &= \tfrac{1}{\pi} \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)} dudv & \\
&= \tfrac{1}{\pi} \textstyle\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta & \text{change to polar coord} \\
&= -\textstyle\int_{r=0}^{\infty} d\left(e^{-r^2}\right) & \text{simplify. Perfect derivative} \\
&= -e^{-r^2}\Big|_0^{\infty} = 1 & \text{integrate \& evaluate}
\end{aligned}$$

(b) Show that $E[X] = a$, so that writing μ in the position occupied by a is a sensible thing to do.

$$\begin{aligned} E[X] &= \tfrac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{\frac{(x-a)^2}{2s^2}} dx & \text{Let } u = \tfrac{x-a}{s\sqrt{2}} \\ &= \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(su\sqrt{2} + a\right) e^{-u^2} du \\ &= \tfrac{s\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} u e^{-u^2} du + \tfrac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} u e^{-u^2} du \end{aligned}$$

The first integrand is an odd function, so the integral is zero. The second was evaluated above. The result is $E[X] = a$. Hence, we can write μ in place of a to represent the position of the mean-value parameter in the expression.

(c) Show that $\text{var}(X) = s^2$ so that writing σ in the position occupied by s is a sensible thing to do.

$$\begin{aligned} \text{var}[X] &= \tfrac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-a)^2 e^{\frac{(x-a)^2}{2s^2}} dx & \text{Let } t = \tfrac{x-a}{s\sqrt{2}} \\ &= \tfrac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2s^2 t^2 e^{-t^2} dt & \text{Integrate by parts} \\ &= \tfrac{-s^2}{\sqrt{\pi}} \left(te^{-t^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-t^2} dt \right) = s^2 \end{aligned}$$

The first term evaluates to zero. The second was evaluated above. Because $\text{var}[X] = s^2$ we are entitled to replace s^2 by the traditional variance symbol, σ^2.

8. Verify the results for the mean and standard deviation of the Rayleigh distribution that are given in Example 3.4.1. Plot the Rayleigh distribution for $b = 1, 4, 9, 25$ and observe the changes in the graph. The Rayleigh distribution is

$$f_R(r) = \frac{r}{b} e^{-\frac{r^2}{2b}}$$

The mean value is

$$\begin{aligned} E[R] &= \int_0^{\infty} \tfrac{r^2}{b} e^{-\frac{r^2}{2b}} dr \\ &= -re^{-\frac{r^2}{2b}} \Big|_0^{\infty} + \int_0^{\infty} e^{-\frac{r^2}{2b}} dr & \text{integration by parts} \\ &= \sqrt{2b} \left(\int_0^{\infty} e^{-t^2} dt \right) & \text{change variable} \end{aligned}$$

The integral equals $\sqrt{\pi}/2$ from Problem 7a. Hence,

$$E[R] = \sqrt{\frac{b\pi}{2}}$$

The mean-squared value can also be computed by using integration by parts.

$$
\begin{aligned}
E[R] &= \int_0^\infty \tfrac{r^3}{b} e^{-\frac{r^2}{2b}} dr \\
&= -r^2 e^{-\frac{r^2}{2b}} \mid_0^\infty + \int_0^\infty 2re^{-\frac{r^2}{2b}} dr \quad \text{integration by parts} \\
&= -2be^{-\frac{r^2}{2b}} \mid_0^\infty = 2b
\end{aligned}
$$

Hence, $\text{var}[R] = E[R^2] - E^2[R] = 2b - \frac{b\pi}{2} \approx 0.43b$

9. Calculate the characteristic function $M_X(j\omega) = E\left[e^{j\omega X}\right]$ for a random variable X that has a Poisson distribution

$$P[X = k] = \frac{\mu^k e^{-\mu}}{k!} \quad k = 0, 1, 2, \ldots$$

Use the characteristic function to compute the moments $E[X]$, $E[X^2]$ and $E\left[X^3\right]$.

By definition, the cf is the sum

$$
\begin{aligned}
M_X(j\omega) &= \sum_{k=0}^{\infty} e^{j\omega k} P(X = k) = \sum_{k=0}^{\infty} \frac{e^{j\omega k}\mu^k e^{-\mu}}{k!} \\
&= e^{-\mu} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mu e^{j\omega}\right)^k
\end{aligned}
$$

We make use of the identity

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Then the characteristic function sums to (using $z = \mu e^{j\omega}$)

$$M_X(j\omega) = e^{-\mu} \exp\left(\mu e^{j\omega}\right)$$

Computing the expected values is now just a matter of taking derivatives.

$$
\begin{aligned}
E[X] &= \frac{1}{j}\frac{d}{d\omega} e^{-\mu} \exp\left(\mu e^{j\omega}\right)\Big|_{\omega=0} = \frac{1}{j} e^{-\mu} j\mu \exp\left(\mu e^{j\omega}\right)\Big|_{\omega=0} \\
&= \frac{1}{j}\, j\mu \exp\left(-\mu + j\omega + \mu e^{j\omega}\right)\big|_{\omega=0} = \mu
\end{aligned}
$$

$$
\begin{aligned}
E\left[X^2\right] &= \frac{1}{j^2}\frac{d^2}{d\omega^2} e^{-\mu} \exp\left(\mu e^{j\omega}\right)\Big|_{\omega=0} \\
&= \frac{1}{j^2}\left[j^2\mu \exp\left(-\mu + j\omega + \mu e^{j\omega}\right) + j^2\mu^2 \exp\left(-\mu + 2j\omega + \mu e^{j\omega}\right)\right]\Big|_{\omega=0} \\
&= \mu + \mu^2
\end{aligned}
$$

$$
\begin{aligned}
E\left[X^3\right] &= \left.\frac{1}{j^3}\frac{d^3}{d\omega^3}e^{-\mu}\exp\left(\mu e^{j\omega}\right)\right|_{\omega=0} \\
&= \frac{1}{j^3}\left[j^3\mu\exp\left(-\mu+j\omega+\mu e^{j\omega}\right)+3j^3\mu^2\exp\left(-\mu+2j\omega+\mu e^{j\omega}\right)+j^3\mu^3\exp\left(-\mu+3j\omega+\mu e^{j\omega}\right)\right]_{\omega=0} \\
&= \mu+3\mu^2+\mu^3
\end{aligned}
$$

In addition, one can compute the variance as

$$
\mathrm{Var}\left(X\right)=E\left[X^2\right]-E^2\left[X\right]=\mu+\mu^2-\mu^2=\mu
$$