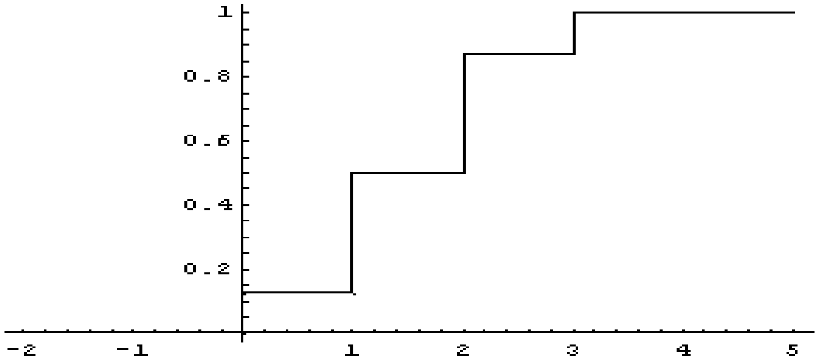
SIMG-713
Homework 2
Solutions

1. Consider the experiment of tossing three fair coins–a penny, a nickel and a dime. Construct a visualization of the sample space $\mathcal{U}$. Let X =Number of Tails. Show the set in $\mathcal{U}$ that corresponds to each value of X. Calculate and plot the distribution function $F_X(x)$.

   The state of each of the three coins is indicated in the table below, where 0 corresponds to H and 1 to T.

k	Penny	Nickle	Dime	N_{heads}
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	2
4	1	0	0	1
5	1	0	1	2
6	1	1	0	2
7	1	1	1	3

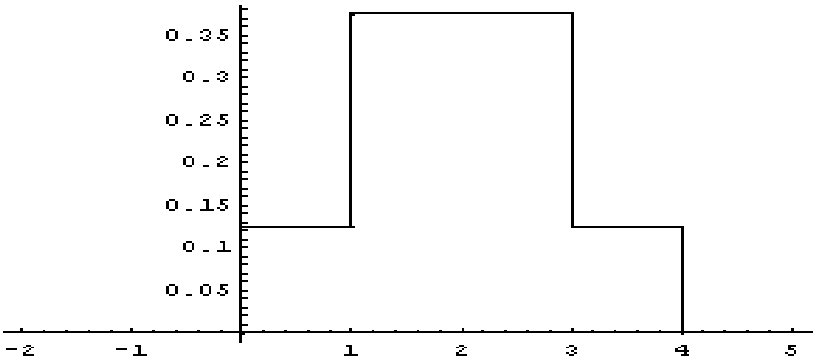
   $X = 0 \Longrightarrow \{e_0\}$, $X = 1 \Longrightarrow \{e_1, e_2, e_4\}$, $X = 2 \Longrightarrow \{e_3, e_5, e_6\}$, $X = 3 \Longrightarrow \{e_7\}$. The cumulative distribution function for X will have steps of size $\{1, 3, 3, 1\}$ at points $\{0, 1, 2, 3\}$. The function is illustrated below.

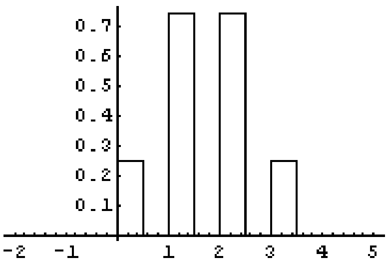
2. Plot the function $D(x) = F_X(x) - F_X(x-1)$ for the distribution function in Exercise 1. Interpret the result in terms of probabilities.

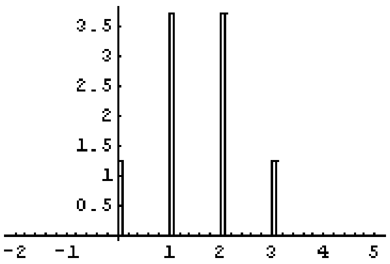
   The function will measure the step sizes. Between the steps it will be constant.

3. Consider the function $d_\Delta(x) = (F_X(x) - F_X(x-\Delta))/\Delta$. Plot this function using the distribution function of problem 1 for $\Delta = 0.5$ and $\Delta = 0.1$. Give an interpretation of the results. What happens as $\Delta \to 0$?



Changing Δ changes the interval over which the difference is taken and scales the height of the difference function. It approaches a derivative as $\Delta \to 0$.





4. Consider the experiment of rolling a fair die. The experiment is:

   Let $\mathbf{E}$ be the experiment of tossing a fair die. Let A be the event that an even face appears and B be the event that the number on the face is four or more. Let $I(S)$ be the indicator function which takes on the value 1 if S is *true* and the value 0 if S is *false.* Then $\mathbf{X} = [I(A), I(B)]$ is a random vector. The points that can be attained by $\mathbf{X}$ are $\{(0,0),(0,1),(1,0),(1,1)\}$, which constitutes the *range* of $\mathbf{X}$. The probability of each point in the range can be found by summing the outcome probabilities in the related events. The event associated with point is a compound event, $(0,0) \Leftrightarrow A^c \cap B^c = \{f_1, f_3\}$, $(0,1) \Leftrightarrow A^c \cap B = \{f_5\}$, $(1,0) \Leftrightarrow A \cap B^c = \{f_2\}$, $(1,1) \Leftrightarrow A \cap B = \{f_4, f_6\}$. The probabilities are $P[\mathbf{X} = (0,0)] = 1/3$, $P[\mathbf{X} = (0,1)] = 1/6$, $P[\mathbf{X} = (1,0)] = 1/6$, $P[\mathbf{X} = (1,1)] = 1/3$.

   For this problem, list the following events and compute their probabilities.

   (a) $X_1 + X_2 = 1$ This happens in events $(0,1)$ and $(1,0)$. The events are mutually exclusive, so that the probability is $P[(0,1)] + P[(1,0)] = 1/6 + 1/6 = 1/3$.

   (b) $X_1 = X_2$ This event includes (0,0) and (1,1). The probability is $P[(0,0)] + P[(1,1)] = 1/3 + 1/3 = 2/3$

(c) $X_1 \neq X_2$ This event includes (0,1) and (1,0). The probability is $P[(0,1)] + P[(1,0)] = 1/6 + 1/6 = 1/3$.

5. Consider the experiment of rolling a pair of fair dice. Let X_1 be the number showing on the first face and X_2 be the number showing on the second. Let $X_3 = X_1 + X_2$. Compute the following probabilities.

   (a) $P[X_3 = 5]$ It is useful to construct a table that shows the individual die values and their sum.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

   Each outcome in the table has probability 1/36. The event $X_3 = 5$ occurs four ways, so the probability is $P[X_3 = 5] = 4/36 = 1/9$

   (b) $P[X_3 = 5 \mid X_1 = 2]$ by using the definition of conditional probability. The joint event $\{X_3 = 5, X_1 = 2\}$ occurs only if $\{X_1 = 2, X_2 = 3\}$ Then $P[X_3 = 5, X_1 = 2] = P[X_1 = 2, X_2 = 3] = 1/36$. Also, $P[X_1 = 2] = 1/6$ Hence,

$$P[X_3 = 5 \mid X_1 = 2] = \frac{P[X_3 = 5, X_1 = 2]}{P[X_1 = 2]} = \frac{1/36}{1/6} = 1/6$$

   (c) $P[X_3 = k]$ for $k = 1, 2, \ldots, 12$. By inspection of the table, using 1/36 for the individual events, we can construct the table below.

k	1	2	3	4	5	6	7	8	9	10	11	12
$P[X_3 = k]$	0	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

   (d) $P[X_3 = k \mid X_1 = 2]$ for $k = 1, 2, \ldots, 12$. First form $P[X_3 = k, X_1 = 2]$ for each k then divide by $P[X_1 = 2] = 1/6$.

k	1	2	3	4	5	6	7	8	9	10	11	12
$P[X_3 = k, X_1 = 2]$	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
$P[X_3 = k \mid X_1 = 2]$	0	0	1/6	1/6	1/6	1/6	1/6	1/6	0	0	0	0

6. Show that the two-dimensional probability density function is non-negative by making use of the definitions in (2.27) and using a definition of the derivative as a limit.
   The 2D distribution function is nondecreasing in moving from any (x, y) to $(x + \Delta x, y + \Delta y)$ where $\Delta x \geq 0$ and $\Delta y \geq 0$. The 2D pdf is formed by taking the difference between the second point and the first and normalizing by dividing by $\Delta x \Delta y$. This is always a nonnegative quantity.

7. Let $\mathbf{W} = (U, V)$ be a two-dimensional random vector with the joint probability density function

$$f_{U,V}(u, v) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp\left[-\frac{u^2 - 2\rho uv + v^2}{2(1 - \rho^2)}\right] \qquad (1)$$

   where $|\rho| \leq 1$. We will learn that this is the joint pdf of correlated normal random variables with correlation coefficient ρ. Show that

$$f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

and

$$f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

for any value of ρ, and determine the value of ρ for which the random variables are statistically independent.
We can find the marginal distribution $f_U(u)$ by integrating over the joint distribution. The technique of completing the square in the exponent and making the change of variable $t = (v - \rho u)/\sqrt{1-\rho^2}$ is used below.

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{UV}(u,v)dv \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{u^2}{2(1-\rho^2)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\rho^2 u^2 - \rho^2 u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right] dv \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{u^2(1-\rho^2)}{2(1-\rho^2)}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right] dv \\
&= \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{t^2}{2}\right] dt\right) \\
&= \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}
\end{aligned}$$

The derivation of $f_V(v)$ is parallel.

8. Suppose that U is uniformly distributed over $[-1, 1]$ and that $V = U^2$. Find $F_V(v)$ and $f_V(v)$. Notice that the transformation is not single-valued so that care must be taken in applying (2.47). For any v in the interval $0 \le v \le 1$ we have

$$\int_0^v f_V(t)dt = \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{2} dt = \sqrt{v}$$

Differentiate both sides with respect to v to obtain

$$f_V(v) = \frac{1}{2\sqrt{v}}$$