Homework 2

1. Consider the experiment of tossing three fair coins–a penny, a nickel and a dime. Construct a visualization of the sample space $\mathcal{U}$. Let X =Number of Tails. Show the set in $\mathcal{U}$ that corresponds to each value of X. Calculate and plot the distribution function $F_X(x)$.

2. Plot the function $D(x) = F_X(x) - F_X(x-1)$ for the distribution function in Exercise 1. Interpret the result in terms of probabilities.

3. Consider the function $d_\Delta(x) = (F_X(x) - F_X(x-\Delta))/\Delta$. Plot this function using the distribution function of problem 1 for $\Delta = 0.5$ and $\Delta = 0.1$. Give an interpretation of the results. What happens as $\Delta \to 0$?

4. Consider the experiment of rolling a fair die as described in Example **??**. List the following events and compute their probabilities

   (a) $X_1 + X_2 = 1$

   (b) $X_1 = X_2$

   (c) $X_1 \neq X_2$

5. Consider the experiment of rolling a pair of fair dice. Let X_1 be the number showing on the first face and X_2 be the number showing on the second. Let $X_3 = X_1 + X_2$. Compute the following probabilities.

   (a) $P[X_3 = 5]$

   (b) $P[X_3 = 5 \mid X_1 = 2]$ by using the definition of conditional probability.

   (c) $P[X_3 = k]$ for $k = 1, 2, \ldots, 12$.

   (d) $P[X_3 = k \mid X_1 = 2]$ for $k = 1, 2, \ldots, 12$.

6. Show that the two-dimensional probability density function is non-negative by making use of the properties of the 2-D probability distribution function.

7. Let $\mathbf{W} = (U, V)$ be a two-dimensional random vector with the joint probability density function

$$f_{U,V}(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right]$$

   where $|\rho| \leq 1$. We will learn that this is the joint pdf of correlated normal random variables with correlation coefficient ρ. Show that

$$f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

and

$$f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

for any value of ρ, and determine the value of ρ for which the random variables are statistically independent.

8. Suppose that U is uniformly distributed over $[-1, 1]$ and that $V = U^2$. Find $F_V(v)$ and $f_V(v)$. Notice that the transformation is not single-valued so that care must be taken in the analysis.