2.5 Exercises

1. Show that two sets $\mathcal{A}$ and $\mathcal{B}$ are equal if and only if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

2. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be events such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$. Show that $P(\mathcal{A}) \leq P(\mathcal{B}) \leq P(\mathcal{C})$.

3. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{A} \subset \mathcal{B}$ if and only if $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$.

4. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{A} \subset \mathcal{B}$ if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

5. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that if $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B}^c \subset \mathcal{A}^c$.

6. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{B} - \mathcal{A} = \mathcal{B} \cap \mathcal{A}^c$.

7. Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots \mathcal{A}_n$ be a partition of $\mathcal{U}$ and let $\mathcal{B}$ any set in $\mathcal{U}$. Let $\mathcal{B}_i = \mathcal{B} \cap \mathcal{A}_i$. Show that $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i$ and $\mathcal{B}_i \cap \mathcal{B}_j = \phi$ if $i \neq j$. Construct an expression for $p(\mathcal{B})$ in terms of the $p(\mathcal{B}_i)$.

8. Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots \mathcal{A}_n$ be arbitrary events in a sample space $\mathcal{U}$. Show that

$$P\left[\bigcup_{i=1}^{n} \mathcal{A}_i\right] \leq \sum_{i=1}^{n} P(\mathcal{A}_i)$$

with equality if and only if the events are mututally exclusive.

9. Prove Theorem 2.3.1 on page 18.

10. We note from Table 2.1 on page 14 that the events $\mathcal{A}_{3,}$ $\mathcal{A}_{12}$, and $\mathcal{A}_{48}$ form a partition of the sample space $\mathcal{U}$ for the die-tossing experiment Use this partition to compute $P(\mathcal{A}_{56})$ using Theorem 2.3.1.

11. Prove Theorem 2.3.2 on page 19.

12. The conditional probability $P(\mathcal{B}|\mathcal{A})$ may be greater than, less than or equal to $P(\mathcal{B})$. Draw a Venn diagram that illustrates each case.

13. Show that if events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent, then so are $\mathcal{A}$ and $\mathcal{B}^c$, $\mathcal{A}^c$ and $\mathcal{B}$, and $\mathcal{A}^c$ and $\mathcal{B}^c$.

14. Show that if events $\mathcal{A}$ and $\mathcal{B}$ with $P(\mathcal{A}) > 0$ and $P(\mathcal{B}) > 0$ are statistically independent then they are not mutually exclusive and vice versa.

15. Show that $P(\mathcal{B}|\mathcal{A}) = P(\mathcal{B})$ implies $P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$.

16. Draw Venn diagrams to illustrate the cases $P(\mathcal{B}|\mathcal{A}) = 1$ and $P(\mathcal{B}|\mathcal{A}) = 0$.

17. Use Bayes' rule to compute $P(\mathcal{A}_{12}|\mathcal{A}_{56})$ for the die-tossing experiment. Note that event $\mathcal{A}_{56}$ corresponds to event $\mathcal{B}$ in Bayes' rule.

18. Repeat the analysis of Example 2.3.1 using $b_1 = 99$ and $b_2 = 101$ with the detector noise distribution

e	-2	-1	0	1	2
$P(e)$	0.1	0.2	0.4	0.2	0.1

and $P(b_1) = 0.6$.

19. Extend the analysis of Example 2.3.1 to a case with five sources of brightness levels $(95, 97, 99, 101, 103)$ assumed to be distributed in equal numbers. Use the detector noise distribution

e	-2	-1	0	1	2
$P(e)$	0.1	0.2	0.4	0.2	0.1