SIMG-713
Homework 1
Solutions

1. Show that two sets $\mathcal{A}$ and $\mathcal{B}$ are equal if and only if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

   Suppose $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$. Then $\mathcal{A} \subset \mathcal{B}$ means $e \in \mathcal{A}$ implies $e \in \mathcal{B}$, so there can be no element in $\mathcal{A}$ that is not in $\mathcal{B}$. Similarly, $\mathcal{B} \subset \mathcal{A}$ says that any element $e \in \mathcal{B}$ is also in $\mathcal{A}$. Hence, the sets contain the same elements. This establishes the "if" part. Now suppose that $\mathcal{A} = \mathcal{B}$. This means that $e \in \mathcal{A}$ implies $e \in \mathcal{B}$, or $\mathcal{A} \subset \mathcal{B}$. It also means that $e \in \mathcal{B}$ implies $e \in \mathcal{A}$, or $\mathcal{B} \subset \mathcal{A}$.

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2. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be events such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$. Show that $P(\mathcal{A}) \leq P(\mathcal{B}) \leq P(\mathcal{C})$.

   Let e_1 be an element of $\mathcal{C}$ that is not in $\mathcal{B}$ (and therefore not in $\mathcal{A}$), and let $e_2 \in \mathcal{B}$ but $\notin \mathcal{A}$. Then, $P(\mathcal{B}) \geq P(\mathcal{A}) + P(e_1)$, with equality if all of the other elements of $\mathcal{B}$ are common ot $\mathcal{A}$. Because $P(e_1) \geq 0$, we must have $P(\mathcal{B}) \geq P(\mathcal{A})$. A similar argument holds for the other inequalities

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3. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{A} \subset \mathcal{B}$ if and only if $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$.

   If $\mathcal{A} \subset \mathcal{B}$ then every element of $\mathcal{A}$ is an element of $\mathcal{B}$, so that $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$. If $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$ then every element of $\mathcal{A}$ is an element of $\mathcal{B}$.

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4. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{A} \subset \mathcal{B}$ if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$.

   If $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$ then every element of $\mathcal{A}$ is an element of $\mathcal{B}$ and $\mathcal{A} \subset \mathcal{B}$. If $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{A} = \mathcal{A} \cap \mathcal{A} \subset \mathcal{A} \cap \mathcal{B}$.

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5. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that if $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B}^c \subset \mathcal{A}^c$.

   Assume there is an element $e \in \mathcal{B}^c$ that is not in $\mathcal{A}^c$. If $e \notin \mathcal{A}^c$ then $e \in \mathcal{A} \Longrightarrow e \in \mathcal{B} \Longrightarrow e \notin \mathcal{B}^c$, which is a contradiction.

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6. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary events. Show that $\mathcal{B} - \mathcal{A} = \mathcal{B} \cap \mathcal{A}^c$.

   Let $e \in (\mathcal{B} - \mathcal{A}) \Longrightarrow e \in \mathcal{B}$ and $e \notin \mathcal{A} \Longrightarrow e \in \mathcal{A}^c \Longrightarrow e \in \mathcal{B} \cap \mathcal{A}^c$. This shows that $(\mathcal{B} - \mathcal{A}) \subset \mathcal{B} \cap \mathcal{A}^c$. To show the opposite, assume $f \in \mathcal{B} \cap \mathcal{A}^c$. Then $f \in \mathcal{B}$ and $f \notin \mathcal{A} \Longrightarrow f \in (\mathcal{B} - \mathcal{A})$, which shows that $\mathcal{B} \cap \mathcal{A}^c \subset \mathcal{B} - \mathcal{A}$. Taken together, this implies $\mathcal{B} - \mathcal{A} = \mathcal{B} \cap \mathcal{A}^c$.

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7. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n$ be a partition of $\mathcal{U}$ and let $\mathcal{B}$ any set in $\mathcal{U}$. Let $\mathcal{B}_i = \mathcal{B} \cap \mathcal{A}_i$. Show that $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i$ and $\mathcal{B}_i \cap \mathcal{B}_j = \phi$ if $i \neq j$. Construct an expression for $p(\mathcal{B})$ in terms of the $p(\mathcal{B}_i)$.

   For any $i \neq j$, $\mathcal{A}_i \cap \mathcal{A}_j = \phi$. Let $\mathcal{B}_i = \mathcal{B} \cap \mathcal{A}_i$ and $\mathcal{B}_j = \mathcal{B} \cap \mathcal{A}_j$. Then $\mathcal{B}_i \cap \mathcal{B}_j = \mathcal{B} \cap \mathcal{A}_i \cap \mathcal{A}_j = \phi$. Let $e \in \mathcal{B}$. Then, for some k, $e \in \mathcal{A}_k \Longrightarrow e \in \mathcal{B}_k \Longrightarrow e \in \bigcup_{i=1}^n \mathcal{B}_i$. Similarly, assume $f \in \bigcup_{i=1}^n \mathcal{B}_i$. Then there is some set $\mathcal{B}_m$ with $e \in \mathcal{B}_m = \mathcal{B} \cap \mathcal{A}_m \Longrightarrow e \in \mathcal{B}$. This proves that $\mathcal{B} = \bigcup_{i=1}^n \mathcal{B}_i$. Since all of the $\mathcal{B}_i$ are disjoint, $P(\mathcal{B}_i \cap \mathcal{B}_j) = P(\phi) = 0$ so that $P(\mathcal{B}) = \sum_{i=1}^n P(\mathcal{B}_i)$.

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8. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n$ be arbitrary events in a sample space $\mathcal{U}$. Show that

$$P\left[\bigcup_{i=1}^n \mathcal{A}_i\right] \leq \sum_{i=1}^n P(\mathcal{A}_i)$$

with equality if and only if the events are mututally exclusive.

For any set $\mathcal{S}$, $P(\mathcal{S})$ is the sum of the probabilities of the outcomes contained in $\mathcal{S}$. Let $\mathcal{S} = \bigcup\limits_{i=1}^{n} \mathcal{A}_i$. Then every outcome is in some $\mathcal{A}_i$ and gets counted on both sides of the above equation. If some event is two or more of the sets, then it is counted two or more times on the right but only once on the left. Because probabilities are non-negative, this establishes the inequality. Equality will therefore only if there is no outcome that is in more than one set, which is the definition of *mutually exclusive*.

9. Prove the following theorem: Suppose events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ partition a sample space $\mathcal{U}$, and that $P(\mathcal{A}_i) > 0$ for all $i = 1, 2, \ldots, n$. Then for any event $\mathcal{B}$ in $\mathcal{U}$

$$P(\mathcal{B}) = \sum_{i=1}^{n} P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i).$$

$\mathcal{B} = \bigcup_{i=1}^{n} (\mathcal{B} \cap \mathcal{A}_i) = \bigcup_{i=1}^{n} \mathcal{B}_i$ where the $\mathcal{B}_i$ are disjoint. By the previous problem

$$P(\mathcal{B}) = \sum_{i=1}^{n} P(\mathcal{B} \cap \mathcal{A}_i).$$

The desired result is obtained by noting that $P(\mathcal{B} \cap \mathcal{A}_i) = P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i)$.

10. We note from Table 2.1 on page 14 that the events $\mathcal{A}_3$, $\mathcal{A}_{12}$, and $\mathcal{A}_{48}$ form a partition of the sample space $\mathcal{U}$ for the die-tossing experiment Use this partition to compute $P(\mathcal{A}_{56})$.

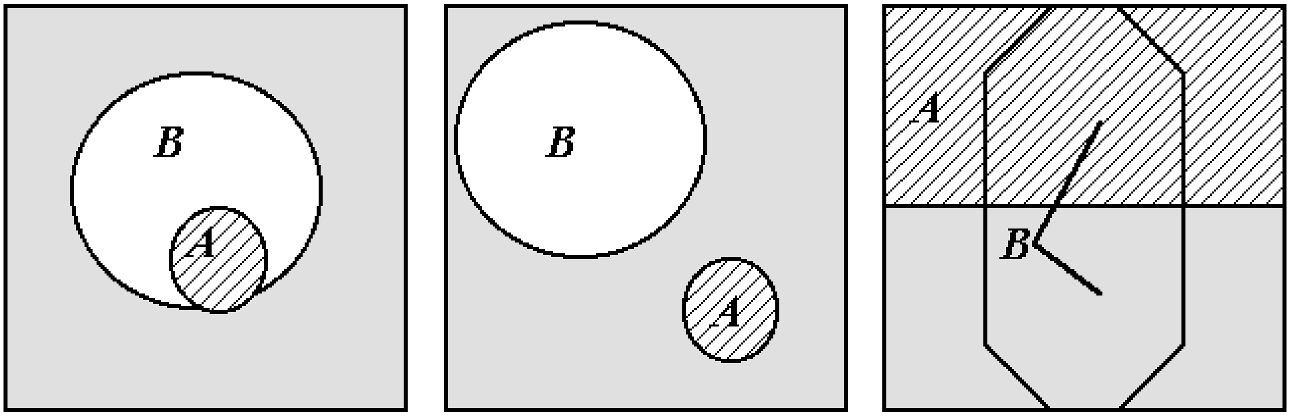
It is convenient to use binary notation to denote set membership. Let $B_3 = A_3 \cap A_{56} = (110000) \cap (000111) = (000000)$ and $P(B_3) = P(\phi) = 0$. $B_{12} = A_{12} \cap A_{56} = (001100) \cap (000111) = (000100) = \{e_4\}$ and $P(B_{12}) = P(e_4) = 1/6$. $B_{48} = A_{48} \cap A_{56} = (000011) \cap (000111) = (000011) = \{e_5, e_6\}$ and $P(B_{48}) = 1/3$. Therefore, $P(A_{56}) = 0 + 1/6 + 1/3 = 1/2$.

11. Prove Bayes' rule..

By the definition of conditional probability, $P(A_j \cap B) = P(A_j)P(B|A_j) = P(B)P(A_j|B)$. Now write $P(\mathcal{B}) = \sum_{i=1}^{n} P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i)$ as in problem 9. Then

$$P(A_j|B) = \frac{P(A_j)P(B|A_j)}{P(B)} = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^{n} P(\mathcal{B}|\mathcal{A}_i)P(\mathcal{A}_i)}$$

12. The conditional probability $P(\mathcal{B}|\mathcal{A})$ may be greater than, less than or equal to $P(\mathcal{B})$. Draw a Venn diagram that illustrates each case.



Left, $P(B|A)$ increases. Center, $P(B|A)$ decreases. Right, $P(B|A)$ remains the same, at about 0.5.

13. Show that if events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent, then so are $\mathcal{A}$ and $\mathcal{B}^c$, $\mathcal{A}^c$ and $\mathcal{B}$, and $\mathcal{A}^c$ and $\mathcal{B}^c$.

    By assumption, $P(A \cap B) = P(A)P(B)$. Since B and B^c form a partition, $A = (A \cap B) \cup (A \cap B^c)$. Therefore $P(A) = P(A \cap B) + P(A \cap B^c)$. Apply the independence assumption to the first term on the right. Then $P(A) = P(A)P(B) + P(A \cap B^c)$. Now, simply solve for $P(A \cap B^c) = P(A)[1 - P(B)] = P(A)P(B^c)$. This proves that A and B^c are statistically independent. The other combinations are proved in a similar fashion.

14. Show that if events $\mathcal{A}$ and $\mathcal{B}$ with $P(\mathcal{A}) > 0$ and $P(\mathcal{B}) > 0$ are statistically independent then they are not mutually exclusive and vice versa.

    If A and B are s.i. Then $P(A \cap B) = P(A)P(B)$. Express the probability of the union as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Make use of the s.i. assumption: $P(A \cup B) = P(A) + P(B) - P(A)P(B)$. The only way that we can have $P(A \cup B) = P(A) + P(B)$ is if either $P(A) = 0$ or $P(B) = 0$, which violates the theorem prescription. Similarly, if A and B are mutually exclusive then $P(A \cup B) = P(A) + P(B)$ so that $P(A \cap B) = 0$, and it is impossible to have $P(A \cap B) = P(A)P(B)$ without either $P(A) = 0$ or $P(B) = 0$.

15. Show that $P(\mathcal{B}|\mathcal{A}) = P(\mathcal{B})$ implies $P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$.

    $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$. Now, use the assumption on $P(B|A)$ so that $P(A \cap B) = P(A)P(B) = P(B)P(A|B)$. Therefore, $P(A|B) = P(A)$.

16. Draw Venn diagrams to illustrate the cases $P(\mathcal{B}|\mathcal{A}) = 1$ and $P(\mathcal{B}|\mathcal{A}) = 0$.

    See the left and center diagrams for problem 12.

17. Use Bayes' rule to compute $P(\mathcal{A}_{12}|\mathcal{A}_{56})$ for the die-tossing experiment. Note that event $\mathcal{A}_{56}$ corresponds to event $\mathcal{B}$ in Bayes' rule.

    $A_{12} = \{e_3, e_4\}$, $A_{56} = \{e_4, e_5, e_6\}$.

    $$P(A_{12}|A_{56}) = \frac{P(A_{12} \cap A_{56})}{P(A_{56})} = \frac{P(e_4)}{P(A_{56})} = \frac{1/6}{1/3} = \frac{1}{3}$$

18. Repeat the analysis of Example 2.3.1 using $b_1 = 99$ and $b_2 = 101$ with the detector noise distribution

e	-2	-1	0	1	2
$P(e)$	0.1	0.2	0.4	0.2	0.1

    and $P(b_1) = 0.6$.

x	97	98	99	100	101	102	103
$P(x\|b_1)$	.1	.2	.4	.2	.1		
$P(x\|b_2)$			.1	.2	.4	.2	.1
$P(x \cap b_1)$	**.06**	**.12**	**.24**	**.12**	.06		
$P(x \cap b_2)$			.04	.08	**.16**	**.08**	**.04**
$P(x)$	.06	.12	.28	.20	.22	.08	.04
$P(b_1\|x)$	**1**	**1**	**.857**	**.6**	.273	0	0
$P(b_2\|x)$	0	0	.143	.4	**.727**	**1**	**1**

    $P(C) = .06 + .12 + .24 + .12 + .16 + .08 + .04 = 0.82$

19. Extend the analysis of Example 2.3.1 to a case with five sources of brightness levels $(95, 97, 99, 101, 103)$ assumed to be distributed in equal numbers. Use the detector noise distribution

e	-2	-1	0	1	2
$P(e)$	0.1	0.2	0.4	0.2	0.1

x	93	94	95	96	97	98	99	100	101	102	103	104	105
$P(x\|b_1)$	.1	.2	.4	.2	.1								
$P(x\|b_2)$			.1	.2	.4	.2	.1						
$P(x\|b_3)$					.1	.2	.4	.2	.1				
$P(x\|b_4)$							.1	.2	.4	.2	.1		
$P(x\|b_5)$									.1	.2	.4	.2	.1
$P(x \cap b_1)$	**.02**	**.04**	**.08**	.04	.02								
$P(x \cap b_2)$			.02	**.04**	**.08**	**.04**	.02						
$P(x \cap b_3)$					.02	.04	**.08**	**.04**	.02				
$P(x \cap b_4)$							.02	.04	**.08**	**.04**	.02		
$P(x \cap b_5)$									.02	.04	**.08**	**.04**	**.02**
$P(x)$	.02	.04	.1	.08	.12	.08	.12	.08	.12	.08	.10	.04	.02
$P(b_1\|x)$	**1**	**1**	**.8**	.5	.167								
$P(b_2\|x)$			.2	**.5**	**.667**	**.5**	.167						
$P(b_3\|x)$					.166	.5	**.667**	**.5**	.167				
$P(b_4\|x)$							.166	.5	**.667**	**.5**	.2		
$P(b_5\|x)$									.166	.5	**.8**	**1**	**1**

$$P(C) = .02 + .04 + .08 + .04 + .08 + .04 + .08 + .04 + .08 + .04 + .08 + .04 + .02 = 0.68$$