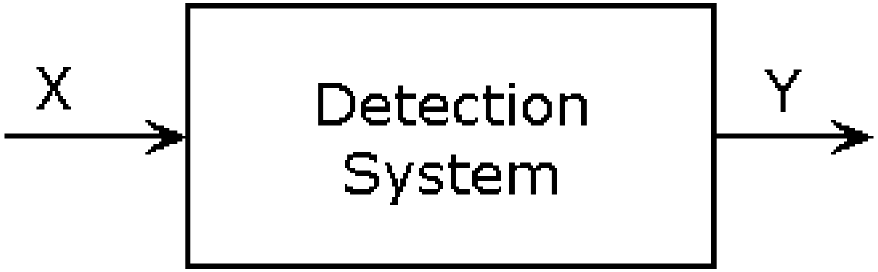
Detective Quantum Efficiency

Lecture 9

Spring 2002

Detector



An input photon stream X is presented to the detector.

X is a random variable with a Poisson distribution.

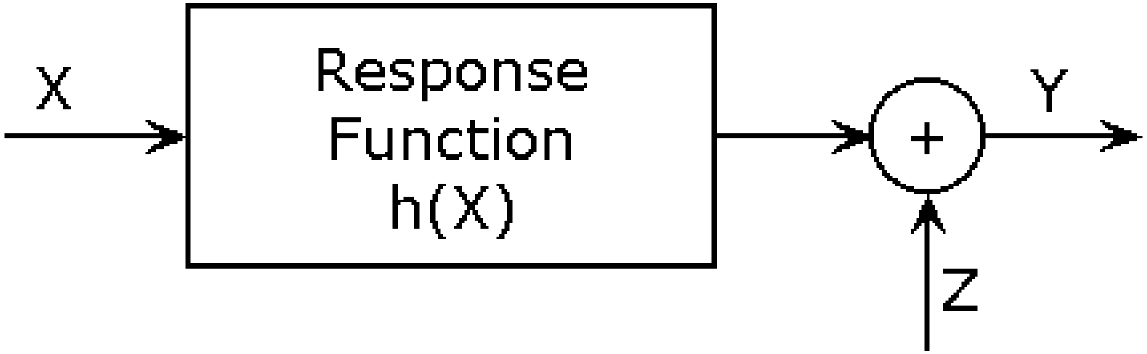
$$P(X = k) = \frac{q^k}{k!} e^{-q} \text{ where } q = \lambda A \tau$$

The response is an event stream Y.

$$Y = h(X) + Z$$

Goal: Determine the intensity λ of the input photon stream.

Detector Model



$$Y = h(X) + Z$$

$h(x)$ is a known detector response function

Z is an independent random variable with a known probability distribution.

Questions:

1. How can we estimate q or λ based on the observation Y?
2. How do we describe the quality of the estimate?

Estimator

An estimator is a rule to calculate the value of a quantity of interest from one or more observations of a random variable.

An estimate is a function of a random variable, and is therefore itself a random variable.

Let $\tilde{\lambda}$ be an estimate of λ. We would like to have $E[\tilde{\lambda}] = \lambda$. This is an *unbiased estimator* of λ.

We would also like to have var$[\tilde{\lambda}]$ be as small as possible.

The quality of the detector is measured by comparing the quality of the estimate that could be made from X to the quality of the estimate that can be made from Y.

Detective Quantum Efficiency

DQE can be defined in a number of equivalent forms. The basic definition that we will use is–

Definition: The DQE for a detector is the ratio of the variance of an estimate of $\hat{\lambda}$ based on the detector input X to the variance of an estimate of $\bar{\lambda}$ based on the detector output Y.

$$DQE = \frac{\text{var}[\hat{\lambda}]}{\text{var}[\bar{\lambda}]}$$

The DQE will always be in the range 0 to 1.

Estimate based on X

Pretend that X is available and construct an unbiased estimate of λ.

The mean value of the input Poisson distribution is

$$E[X] = q = \lambda A\tau \Rightarrow \lambda = \frac{E[X]}{A\tau}$$

Given an observation X, construct an estimate

$$\hat{\lambda} = \frac{X}{A\tau}$$

This estimator is unbiased, since

$$E[\hat{\lambda}] = \frac{E[X]}{A\tau} = \frac{q}{A\tau} = \lambda$$

The variance of the estimator is

$$\text{var}[\hat{\lambda}] = \frac{\text{var}[X]}{A^2\tau^2} = \frac{q}{A^2\tau^2} = \frac{\lambda}{A\tau}$$

Estimate based on Y

To estimate q or λ based on Y we need an equation of the form

$$\bar{q} = w(Y)$$

To find this function, begin with the inverse function and then invert.

From a previous lecture,

$$E[Y] = \mu_Y(q) = L\left(1 - f_1(q, T, S)e^{-q}\right)$$

Given any chosen "operating point" q_0 we can write an expression for $\mu_Y(q)$ in terms of $\mu_Y(q_0)$.

$$\mu_Y(q) = \mu_Y(q_0) + \left.\frac{d\mu_Y}{dq}\right|_{q_0}(q - q_0) + \cdots$$

Estimate (continued)

This gives us a hint at the estimator equation. Replace $\mu_Y(q)$ by Y and q by the estimate $\bar{q}$ and drop higher order terms.

$$Y = \mu_Y(q_0) + \left.\frac{d\mu_Y}{dq}\right|_{q_0} (\bar{q} - q_0)$$

The derivative is the detector gain at the operating point. Solve for the estimate as a function of Y

$$\bar{q} = q_0 + \frac{Y - \mu_Y(q_0)}{g} = w(Y)$$

The function $\bar{q} = w(Y)$ is an unbiased estimator of q near $q = q_0$.

An unbiased estimate of λ is

$$\bar{\lambda} = \frac{\bar{q}}{A\tau} = \frac{w(Y)}{A\tau g}$$

Estimator (continued)

The variance of $\bar{\lambda}$ is

$$\text{var}[\bar{\lambda}] = \frac{\text{var}[\text{Y}]}{(A\tau g)^2}$$

From the results of the last lecture, the variance of a piecewise linear detector is

$$\text{var}[\text{Y}] = L^2 \left[(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2 \right]$$

The gain is given by

$$g(q_0) = \left. \frac{d\mu_Y}{dq} \right|_{q=q_0} = L e^{-q_0} f_2(L, q_0)$$

where

$$f_2(L, q) = \frac{1}{L} \sum_{k=0}^{L-1} \frac{q^k}{k!}$$

DQE

The DQE of a piecewise linear detector can be calculated by

$$
\begin{aligned}
DQE &= \frac{\text{var}[\widehat{\lambda}]}{\text{var}[\bar{\lambda}]} \\
&= \frac{(A\tau g)^2\lambda/A\tau}{L^2\left[(1-f_3e^{-q})-(1-f_1e^{-q})^2\right]} \\
&= \frac{g^2q}{L^2\left[(1-f_3e^{-q})-(1-f_1e^{-q})^2\right]}
\end{aligned}
$$

This is the same result as that obtained in the last lecture for the *comparative noise level* when the variance of Z is zero.

Example–Linear Amplifier

Consider a system in which $Y = gX + Z$, where X is a Poisson r.v. with mean $q = \lambda A\tau$, g is known, and Z is an independent r.v. with zero mean and variance σ_Z^2.

An estimate of λ based on X would be

$$\hat{\lambda} = \frac{X}{A\tau}$$

The estimate has variance

$$\text{var}[\hat{\lambda}] = \frac{\text{var}[X]}{A^2\tau^2} = \frac{\lambda}{A\tau}$$

An estimate based on Y is

$$\bar{\lambda} = \frac{Y}{A\tau g}$$

Linear Amplifier (continued)

This estimate has variance

$$\text{var}[\bar{\lambda}] = \frac{\text{var}[Y]}{A^2\tau^2 g^2} = \frac{g^2\sigma_X^2 + \sigma_Z^2}{A^2\tau^2 g^2} = \frac{\lambda}{A\tau} + \frac{\sigma_Z^2}{A^2\tau^2 g^2}$$

The DQE is

$$\begin{aligned} DQE &= \frac{\text{var}[\hat{\lambda}]}{\text{var}[\bar{\lambda}]} = \frac{\frac{\lambda}{A\tau}}{\frac{\lambda}{A\tau} + \frac{\sigma_Z^2}{A^2\tau^2 g^2}} \\ &= \frac{A\tau\lambda g^2}{A\tau\lambda g^2 + \sigma_Z^2} = \frac{A\tau\lambda}{A\tau\lambda + \frac{\sigma_Z^2}{g^2}} = \frac{1}{1 + \frac{\sigma_z^2}{g^2 q}} \end{aligned}$$

The DQE can be viewed from either the input or output perspective.

The DQE for a linear amplifier is close to unity when σ_Z/g is small compared to $q = A\tau\lambda$. Increasing g improves the DQE.

Example–Secondary Emission

Each input photon produces an output event with probability η. If $q_X = \lambda A\tau$ then $q_Y = \eta\lambda A\tau$ and $E[Y] = \eta\lambda A\tau$

The output process is still Poisson, so that

$$\text{var}(Y) = E[Y] = \eta\lambda A\tau$$

The gain is $g = \eta A\tau$.

The ideal estimate is $\hat{\lambda} = X/A\tau$ so that

$$\text{var}[\hat{\lambda}] = \text{var}[X]/A^2\tau^2 = \lambda/A\tau$$

Then

$$DQE = \frac{var(\hat{\lambda})}{var(\bar{\lambda})} = \frac{\lambda/A\tau}{\eta\lambda A\tau}(\eta A\tau)^2 = \eta$$

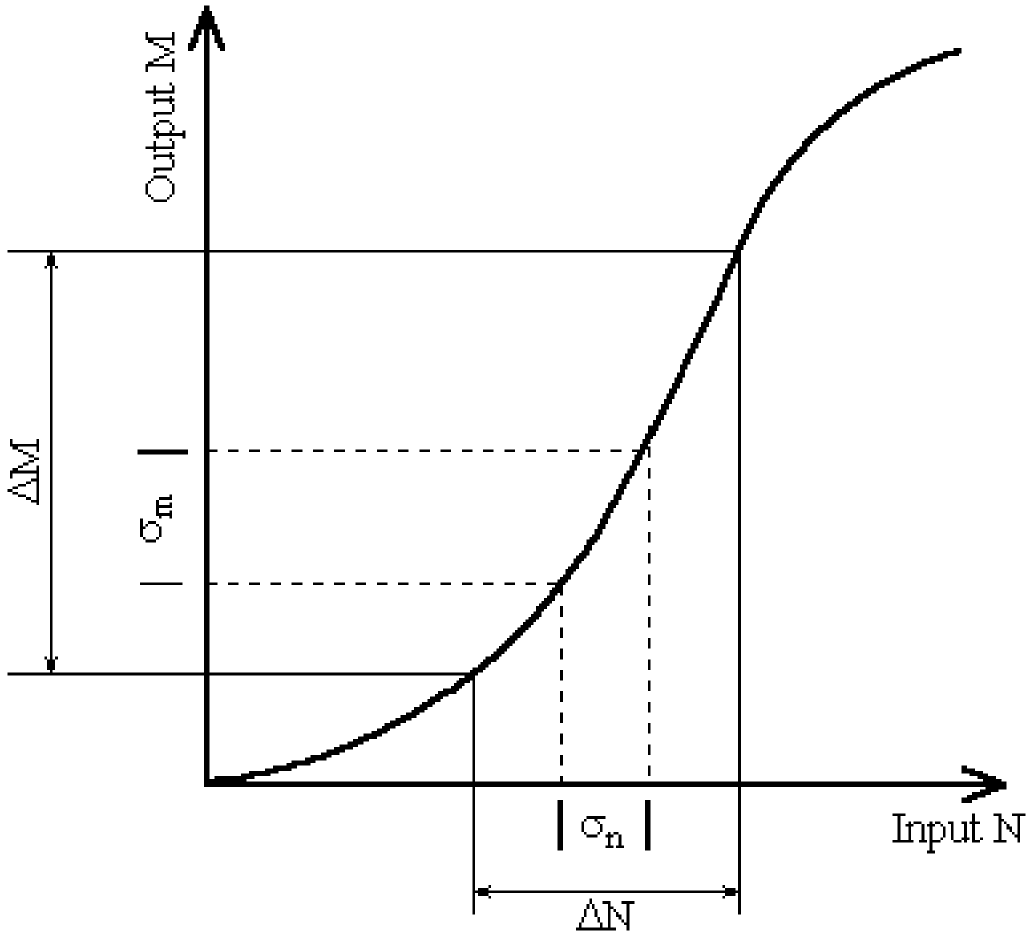
DQE and SNR

DQE may also be defined as the ratio of the (power) SNR at the detector output to the maximum possible SNR.

$$\text{SNR}_{in} = \frac{\Delta N}{\sigma_n}$$

$$\text{SNR}_{out} = \frac{\Delta M}{\sigma_m}$$

$$DQE = \frac{(\text{SNR}_{out})^2}{(\text{SNR}_{in})^2} = \frac{(\Delta M)^2/\sigma_m^2}{(\Delta N)^2/\sigma_n^2}$$



Linear Amplifier Revisited

Input Signal$=E[N] = q$
Input Noise$=\sigma_n = \sqrt{q}$
Output Signal$=E[M] = gq$
Output Noise$=\sigma_m = \sqrt{g^2 q + \sigma_z^2}$

$$
\begin{aligned}
DQE &= \frac{(\Delta M)^2/\sigma_m^2}{(\Delta N)^2/\sigma_n^2} \\
&= \frac{(gq)^2/(g^2 q + \sigma_z^2)}{(q/\sqrt{q})^2} \\
&= \frac{g^2 q}{g^2 q + \sigma_z^2} = \frac{1}{1 + \frac{\sigma_z^2}{g^2 q}}
\end{aligned}
$$

which is equivalent to our previous result.

Noisy Detector

A photon detector has a dark noise whose variance $\sigma_D^2 = 0.1$ counts/second is proportional to the exposure time and a readout noise whose variance $\sigma_R^2 = 10$ counts is at a fixed level. The detector area is $A = 0.3$ cm^2 and the detector conversion efficiency is $\epsilon = 0.75$ counts/photon.

Find the DQE for when the detector is placed in a flux of $\Phi = 3.3$ photons/cm^2/s and exposed for $T = 10$ minutes.

Input Signal & Noise

The input signal is the expected number of photons to the detector. This is

$$S_{in} = q = \Phi AT = (3.3)(0.3)(10)(60) = 594 \text{ photons}$$

The standard deviation of the photon stream is the input noise.

$$N_{in} = \sqrt{q} = 24.4 \text{ photons}$$

The input SNR is

$$\left(\frac{S}{N}\right)_{in} = \frac{594}{24.4} = 24.4$$

Output Signal & Noise

The output signal is the input signal times the conversion efficiency

$$S_{out} = q\epsilon = 5940.75 = 445 \text{ counts}$$

The output noise has three contributors–the Poisson noise in the detected counts, the dark noise and the readout noise.

$$\sigma^2_{N_{out}} = q\epsilon + \sigma^2_D T + \sigma^2_R = 445 + 0.1 \times 600 + 10^2 = 615$$

The standard deviation of the output noise is

$$\sigma_{N_{out}} = 24.8 \text{ counts}$$

The output SNR is then

$$\left(\frac{S}{N}\right)_{out} = \frac{445}{24.8} = 18$$

DQE of Noisy Detector

The DQE can be computed by

$$DQE = \left[\frac{(S/N)_{out}}{(S/N)_{in}}\right]^2 = \left(\frac{18}{24.4}\right)^2 = 0.54$$