Chapter 10

Image Formation in the Ray Model

We know that light rays are deviated at interfaces between media with different refractive indices. The goal in this section is to use interfaces of specified shapes to “collect” and “redirect” rays (or to “collect” and “reshape” wavefronts that are normal to the family of rays) in such a way to create “images” of the original source(s).

10.1 Refraction at a Spherical Surface

Consider a point source located at \( o \). The distance to the vertex \( v \) is \( ov \equiv s_1 > 0 \) as drawn. The distance from vertex \( v \) to the point \( p \) is \( vp \equiv s_2 > 0 \). The distance traveled by a ray in medium \( n_1 \) to the surface is \( va \equiv \ell_1 \) and the distance in medium \( n_2 \) is \( ap \equiv \ell_2 \). The radius of curvature of the surface is \( ve = ac \equiv R > 0 \). For emphasis, we repeat that \( s_1, s_2, \) and \( R \) are all positive in our convention. The ray intersects the surface at the “position angle” \( \phi \) measured from the center of curvature.
c to a. The optical path length of the ray from o to p that passes through a is

\[ OPL = n_1 \ell_1 + n_2 \ell_2 \]

We will evaluate this length by considering the triangles \( \triangle oac \) and \( \triangle acp \); the hypotenuses \( oa \) and \( ap \) may be evaluated by applying the law of cosines:

\[ \triangle oac \implies |oa|^2 = |oc|^2 + |ac|^2 - 2|oc||ac| \cos \varphi \]
\[ \ell_1^2 = (s_1 + R)^2 + R^2 - 2R(s_1 + R) \cos \varphi \]
\[ \ell_1 = \sqrt{(s_1 + R)^2 + R^2 - 2R(s_1 + R) \cos \varphi} \]

\[ \triangle acp \implies |ap|^2 = |ac|^2 + |cp|^2 - 2|ac||cp| \cos (\pi - \varphi) \]
\[ \ell_2^2 = (s_2 - R)^2 + R^2 - 2R(s_2 - R) \cos (\pi - \varphi) \]
\[ \ell_2 = \sqrt{(s_2 - R)^2 + R^2 + 2R(s_2 - R) \cos \varphi} \]
\[ = \sqrt{(s_2 - R)^2 + R^2 - 2R(R - s_2) \cos \varphi} \]

Therefore the optical path length is:

\[ OPL = n_1 \ell_1 + n_2 \ell_2 \]
\[ = n_1 \cdot \left( \sqrt{(s_1 + R)^2 + R^2 - 2R(R + s_1) \cos \varphi} \right) \]
\[ + n_2 \cdot \left( \sqrt{(s_2 - R)^2 + R^2 - 2R(R - s_2) \cos \varphi} \right) \]
which is obviously a function of the “position angle” \( \varphi \). We can now apply Fermat’s principle and evaluate the angle \( \varphi \) for which the OPL is a minimum:

\[
\frac{d}{d \varphi} (OPL) = 0
\]

\[
= \frac{n_1 \cdot 2R (R + s_1) \sin \varphi}{\sqrt{(s_1 + R)^2 + R^2 - 2R (R + s_1) \cos \varphi}} + \frac{n_2 \cdot 2R (R - s_2) \sin \varphi}{\sqrt{(s_2 - R)^2 + R^2 - 2R (R - s_2) \cos \varphi}}
\]

\[
= 2R \sin \varphi \frac{n_1 (R + s_1)}{\sqrt{(s_1 + R)^2 + R^2 - 2R (R + s_1) \cos \varphi}} + 2R \sin \varphi \frac{n_2 (R - s_2)}{\sqrt{(s_2 - R)^2 + R^2 - 2R (R - s_2) \cos \varphi}}
\]

\[
= 2R \sin \varphi \left( \frac{n_1 (R + s_1)}{\ell_1} + \frac{n_2 (R - s_2)}{\ell_2} \right)
\]

\[
\Rightarrow \frac{n_1 (R + s_1)}{\ell_1} + \frac{n_2 (R - s_2)}{\ell_2} = 0
\]

\[
\Rightarrow \frac{n_1 R}{\ell_1} + \frac{n_2 R}{\ell_2} = \frac{n_2 s_2}{\ell_2} - \frac{n_1 s_1}{\ell_1}
\]

\[
\Rightarrow \frac{n_1}{\ell_1} + \frac{n_2}{\ell_2} = \frac{1}{R} \left( \frac{n_2 s_2}{\ell_2} - \frac{n_1 s_1}{\ell_1} \right)
\]

for one spherical surface

This relation between the physical path lengths \( \ell_1 \) and \( \ell_2 \) and the distances \( s_1 \) and \( s_2 \) is exact. Now we now identify the ratio of the physical path length \( \ell_1 \) from \( o \) to \( a \) to the axial distance \( s_1 \) from \( o \) to the surface vertex \( v \):

\[
\frac{\ell_1}{s_1} = \frac{\sqrt{(s_1 + R)^2 + R^2 - 2R (s_1 + R) \cos \varphi}}{s_1}
\]

\[
= \left( \frac{(s_1 + R)^2 + R^2 - 2R (s_1 + R) \cos \varphi}{s_1} \right)^{\frac{1}{2}}
\]

\[
= \left( \frac{s_1^2 + R^2 + 2Rs_1 + R^2 - 2R^2 \cos \varphi - 2Rs_1 \cos \varphi}{s_1^2} \right)^{\frac{1}{2}}
\]

\[
= \left( 1 + \left( \frac{2R^2}{s_1^2} + \frac{2R}{s_1} \right) (1 - \cos \varphi) \right)^{\frac{1}{2}}
\]

This relation also is exact.
10.1.1 Paraxial Approximation

We can expand the cosine of the position angle into its Taylor series:

$$\cos[\varphi] = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \cdots$$

If the angle $\varphi$ is very small (so that the ray remains close to the axis), we can approximate the cosine by truncating the series after the first term:

$$\text{if } \varphi \approx 0 \implies \cos[\varphi] \approx 1 \implies 1 - \cos[\varphi] \approx 0$$

$$\implies \ell_1 \approx 1 \implies \ell_1 \approx s_1, \text{ and } \ell_2 \approx s_2$$

In this, the paraxial approximation (the resulting approximate calculations are called first-order optics or Gaussian optics), we have:

$$\frac{1}{R} \left( \frac{n_2 s_2}{\ell_2} - \frac{n_1 s_1}{\ell_1} \right) \approx \frac{1}{R} (n_2 - n_1)$$

$$\implies \frac{n_1}{s_1} + \frac{n_2}{s_2} \approx \frac{1}{R} (n_2 - n_1)$$

paraxial imaging equation for single spherical surface

Snell’s Law in the Paraxial Approximation

Recall Snell’s law that relates the ray angles before and after refraction:

$$n_1 \sin[\theta_1] = n_2 \sin[\theta_2]$$

In the paraxial approximation where $\theta \approx 0$, the refraction equation simplifies to:

$$n_1 \theta_1 = n_2 \theta_2 \implies \frac{\theta_2}{\theta_1} = \frac{n_1}{n_2} \implies \theta_2 = \frac{n_1}{n_2} \theta_1$$

In words, the ray angle after refraction is proportional to that before refraction.

Power

The value of a lens or lens system is due to its ability to “redirect” rays by changing their direction. This capability is described by the power of the lens; a lens system with a large power changes the angles of rays by a large amount. A lens system that does not change the angles of rays has zero power. Most people are more familiar with the concept of “focal length”, which is the reciprocal of the power. The power is measured in terms of the reciprocal length; if measured in meters, the units of power are diopters.
Object- and Image-Space Focal Lengths

Now consider some pairs of object and image distances \( s_1 \) and \( s_2 \). If the object is located at \(-\infty\), then:

\[
\frac{n_1}{\infty} + \frac{n_2}{s_2} = \frac{n_2}{s_2} \approx \frac{1}{R} (n_2 - n_1)
\]

\[\Rightarrow s_2 \approx \frac{n_2 R}{n_2 - n_1} \equiv f_2\]

the “image-space focal length” of the single surface

If the image is located at \(+\infty\), we have:

\[s_1 \approx \frac{n_1 R}{n_2 - n_1} \equiv f_1\]

the “object-space focal length” of the single surface

Also note that:

\[
\frac{f_1}{f_2} = \left( \frac{n_1 R}{n_2 - n_1} \right) = \frac{n_1}{n_2}
\]

In words, the ratio of the object-space and image-space focal lengths of the single surface between two media equals the ratio of the indices of refraction.

To summarize, the assumptions of paraxial optics reduce the exact trigonometric expressions for ray heights and ray angles to zero. The resulting expressions are accurate ONLY within an infinitesimal region centered on the optical axis of symmetry (the imaginary line through the centers of curvature of the surfaces of the system). Though limited in its descriptive accuracy of the properties of the system, the paraxial approximation results in a set of simple equations that are accurate for locating the axial positions of images. However, because extended objects consist of point sources at various positions in space distant from the optical axis (and thus do not fulfill the requirements of the paraxial approximation), the quality of the image cannot be inferred from this description. “Defects” in the image are due to “aberrations” in the optical system, which are deviations from the paraxial approximation where the output angle is proportional to the input angle.

10.1.2 Nature of Objects and Images:

1. **Real Object**: Rays incident on the lens are diverging from the source; the object distance is positive:
2. Virtual Object: Rays converge toward the “source”, which is “behind” the lens; object distance is negative:

3. Real Image: Rays converge from the lens toward the image; image distance is positive:

4. Virtual Image: Rays diverge from the lens, so that the “image” is behind the lens; the image distance is negative:
10.2 Imaging With Lenses

Normally we do not consider the case of an object in one medium with the image in another – usually both object and image are in air and a lens (a “device” with another index and two usually curved surfaces) forms the image. We can derive the formula for the object and image distances if we know the radii of the lens surfaces and the indices of refraction. We merely cascade the paraxial formulas for a single surface:

At first surface: \[ \frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{n_2 - n_1}{R_1} \]

At second surface: \[ \frac{n_2}{s_2} + \frac{n_3}{s'_2} = \frac{n_3 - n_2}{R_2} \]

where \( s_1 \) is the (usually known) object distance, \( s'_1 \) is the image distance for rays refracted by the first surface, \( s_2 \) is the object distance for the second surface, and \( s'_2 \) is the image distance for rays exiting the second surface (and thus from the lens). For the common “convex-convex” lens, the center of curvature of the lens is to the right of the vertex, and thus the radius \( R_1 \) of the first surface is positive. Since the vertex is to the right of the center of curvature of the second surface, \( R_2 < 0 \). If the lens is “thin”, then the ray encounters the second surface immediately after refraction at the first surface, so the magnitude of the image distance for the front surface \(|s'_1|\) is the same as the object distance for the second surface \(|s_2|\). From the directed distance convention, \( s_2 = -s'_1 \). If the lens is “thick”, then \(|s'_1| \neq |s_2|\), so we define the thickness \( t \) to satisfy:

\[ s'_1 + s_2 = t \implies s'_1 = t - s_2 \]

To find the single imaging equation, we add the equations for the two surfaces:

\[ \left( \frac{n_1}{s_1} + \frac{n_2}{s'_1} \right) + \left( \frac{n_2}{s_2} + \frac{n_3}{s'_2} \right) = \left( \frac{n_2 - n_1}{R_1} \right) + \left( \frac{n_3 - n_2}{R_2} \right) \]

\[ = \frac{n_3}{R_2} + n_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{n_1}{R_1} \]

For a thin lens \((t = 0)\), substitute \(-s_2\) for \( s'_1 \)

\[ \left( \frac{n_1}{s_1} + \frac{n_2}{-s_2} \right) + \left( \frac{n_2}{s_2} + \frac{n_3}{s'_2} \right) = \frac{n_1}{s_1} + \frac{n_3}{s'_2} \]

\[ = \frac{n_3}{R_2} + n_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{n_1}{R_1} \]

where the object is in index \( n_1 \), the lens has index \( n_2 \), and the image is in index \( n_3 \).

In the usual case, the object and image are both in air, so that \( n_3 = n_1 = 1 \). The simplified expression of the power of a thin lens is encapsulated in the so-called
lensmaker’s equation:

\[
\frac{1}{s_1} + \frac{1}{s'_2} = \frac{1}{R_2} + n_2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right) - \frac{1}{R_1} \\
\varphi = \frac{1}{f} = (n_2 - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

which defines the focal length of a lens with known index and surface radii. Note that the object distance \(s_1\) and the image distance \(s'_2\) both “appear” on the left side with the same algebraic sign; this may be interpreted as demonstrating an “equivalence” of the object and image. The reversibility of rays implies that the roles of object and image may be exchanged. The object and image may be considered to be maps of each other, with the mapping function defined by the lens. Corresponding object and image points (or object and image lines or object and image planes) are called conjugate points (or lines or planes).

10.2.1 Examples:

1. Plano-convex lens, curved side forward:

\[
R_1 = |R_1| > 0 \\
R_2 = \pm \infty \text{ (sign has no effect)}
\]

\[
\frac{1}{s_1} + \frac{1}{s'_2} = (n_2 - 1) \left( \frac{1}{|R_1|} - \frac{1}{\infty} \right) = \frac{n_2 - 1}{|R_1|} > 0
\]

If \(s_1 = +\infty\), then \(s'_2 = f > 0\), the focal length

\[
\frac{1}{f} = \frac{n_2 - 1}{R_1} = \varphi \text{ the power (measured in m}^{-1} = \text{diopters)}
\]

\[
f = \frac{R_1}{n_2 - 1} \approx 2R_1 \text{ (since } n_2 \approx 1.5 \text{ for glass)}
\]

We often use the “power” \(\varphi = f^{-1}\) to describe the lens, since powers add (simpler than adding reciprocals of focal lengths).

2. Plano-convex lens, plane side forward:

\[
R_1 = \pm \infty \\
R_2 = -|R_2| < 0
\]

\[
\frac{1}{s_1} + \frac{1}{s'_2} = -\frac{(n_2 - 1)}{R_2} = +\left( \frac{n_2 - 1}{|R_2|} \right) > 0
\]

\[
f = \frac{|R_2|}{n_2 - 1} \approx 2 |R_2|
\]
3. Plano-concave, plane side forward:

\[
\begin{align*}
R_1 &= \pm \infty \\
R_2 &= + |R_2| > 0 \\
\frac{1}{s_1} + \frac{1}{s_2'} &= (n_2 - 1) \left( \frac{1}{\infty} - \frac{1}{|R_2|} \right) = -\frac{(n_2 - 1)}{|R_2|} < 0 \\
f &= - \frac{|R_2|}{n_2 - 1} \approx -2 |R_2|
\end{align*}
\]

4. Double convex lens with equal radii:

\[
\begin{align*}
R_1 &= |R| > 0 \\
R_2 &= -R_1 = - |R| \\
\frac{1}{s_1} + \frac{1}{s_2'} &= (n_2 - 1) \left( \frac{1}{|R|} - \left( - \frac{1}{|R|} \right) \right) = 2 \frac{(n_2 - 1)}{|R|} > 0 \\
\frac{1}{f} &= \varphi = \frac{2 \cdot (n_2 - 1)}{|R|} \\
f &= \frac{|R|}{2 \cdot (n_2 - 1)} \approx |R| > 0 \quad (\text{since } n_2 \approx 1.5)
\end{align*}
\]

\[f \approx |R| \text{ for an equiconvex glass lens}\]

10.3 Magnifications

The most common use for a lens is to change the apparent size of an object (or image) via the magnifying properties of the lens. The mapping of object space to image space “distorts” the size and shape of the image, i.e., some regions of the image are larger and some are smaller than the original object. We can define two types of magnification: transverse and longitudinal.

10.3.1 Transverse Magnification:

The transverse magnification \(M_T\) is what we usually think of as magnification – it is the ratio of object to image dimension measured transverse to the optical axis:
From the similar triangles \( \triangle a_1b_1c \) and \( \triangle a_2b_2c \), we see that:

\[
\frac{y}{s_1} = \frac{|y_2|}{s_2} = -\frac{y_2}{s_2} \quad \text{because } y_2 < 0
\]

\[
\Rightarrow \frac{y_2}{y_1} = -\frac{s_2}{s_1} \equiv M_T
\]

if \( |M_T| > 1 \), the image is \textit{magnified}

if \( |M_T| < 1 \), the image is \textit{minified}

\( M_T < 0 \Rightarrow \) the image is \textit{inverted}

\( M_T > 0 \Rightarrow \) the image is \textit{upright} (or \textit{erect})

Consider the case where the object is located \textit{at} the lens, so that \( s_1 = 0 \). From the imaging equation, we can find \( s_2 \):

\[
s_2 = \left( \frac{1}{f} - \frac{1}{0} \right)^{-1} \Rightarrow s_2 = 0
\]

In words, the image distance also is 0. The transverse magnification is not well defined from the equation, but the distances show that object, lens, and image all coincide, which leads to the observation that \( M_T = +1 \).

10.3.2 Longitudinal Magnification:

The longitudinal magnification \( M_L \) is the ratio of the \textit{length of the image along the optical axis} to the corresponding \textit{length of the object}. Since the inverse distances are related in the paraxial imaging equation \( (s_1^{-1} \text{ and } s_2^{-1}) \), the longitudinal magnification varies for different object distances. The longitudinal magnification is the ratio of differential (infinitesimal) elements of length of the image and object:

\[
M_L = \frac{\Delta s_2}{\Delta s_1} \rightarrow \frac{ds_2}{ds_1}
\]

If evaluated at a single on-axis point (so that \( \Delta s_1 \rightarrow 0 \)), then the infinitesimal quantities are related and the longitudinal magnification is the derivative of the image size relative to the object size:

\[
M_L|_{s_1} = \frac{ds_2}{ds_1}
\]
The expression may be derived by evaluating the total derivative of the object and image distances:

\[ \frac{1}{s_1} + \frac{1}{s_2} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \]

\[ d \left( \frac{1}{s_1} + \frac{1}{s_2} \right) = d \left( \frac{1}{s_1} \right) + d \left( \frac{1}{s_2} \right) = 0 \text{ (because the terms on the RHS are constants)} \]

\[ \implies \left( -\frac{1}{s_1^2} \right) ds_1 + \left( -\frac{1}{s_2^2} \right) ds_2 = 0 \]

\[ \implies M_L = \frac{ds_2}{ds_1} = -\left( \frac{s_2}{s_1} \right)^2 = -(M_T)^2 < 0 \]

The longitudinal magnification of a positive lens is negative because the image moves away from the lens (increasing \( s_2 \)) as the object moves towards the lens (decreasing \( s_1 \)). The longitudinal magnification also affects the irradiance of the image (i.e., the “flux density” of the rays at the image). If \(|M_L|\) is large, then the light in the vicinity of an on-axis location is “spread out” over a larger region of space at the image, so the irradiance of the image is decreased.

The scaling of the 3-D “image” along the three axes. The scaling along the “transverse” axes \( x \) and \( y \) define the transverse magnification, while the scaling of the image along the \( z \)-axis is determined by the longitudinal magnification.
The concept of longitudinal magnification applied to a luminous rod. The section located at \( s_1 = 2f \) is imaged with unit transverse magnification at \( s_2 = 2f \). Sections of the rod with \( s_1 > 2f \) are imaged “closer” to the lens (\( s_2 < 2f \)), and the the energy density is remapped to account for the nonlinear distance relationship

\[
\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}.
\]

10.4 Spherical Mirrors

Consider a mirror with a spherical reflective surface. We can define a power of the surface that is analogous to the power (and thus related to the focal length) of a thin lens. A ray that passes through the center of curvature of the spherical mirror must intersect the surface at right angles, so that the incident angle measured relative to the normal is \( \theta = 0 \). The reflected angle also is \( \theta = 0 \) and thus the reflected ray also passes through the center of curvature. A ray that crosses the optical axis at \( O \) such that \( OC \cong 0 \) makes an angle of \( -\theta \cong 0 \) with the surface normal. It therefore is reflected at an angle \( +\theta \) measured relative to the normal and intersects the axis at \( O' \) such that \( O'C \cong -OC \cong 0 \). A paraxial ray from an object at \( \infty \) makes a small angle \( -\theta \cong 0 \) and reflects at \( +\theta \cong 0 \). The reflected paraxial ray intersects the optical axis at a distance \( VF' \cong CV \cdot \frac{1}{2} = \frac{R}{2} \). From the triangle

Ray redirection by a spherical mirror: (a) an incoming ray through the center of curvature \( C \) produces an exiting ray that also passes through \( C \) because the ray angles are \( 0 \); (b) a ray that just misses \( C \) intersects the mirror at a ray angle of \( -\theta \) and produces a ray at an angle of \( +\theta \) that misses \( C \) to the opposite side; (c) a paraxial ray from an object at infinity intersects the surface at an angle \( -\theta \cong 0 \) and
is reflected at $+\theta$ to cross the optical axis at $F'$, which is halfway between $C$ and $V$, which shows that $f = -\frac{R}{2}$ for paraxial rays.

Now consider the case where a nonparaxial ray from an object at $\infty$ reaches the mirror and makes a large angle measured from the surface normal. The ray is reflected at the same large angle $\theta$ and intersects the optical axis at $O'$ as shown.

Ray diagram that illustrates spherical aberration of a spherical mirror. The nonparaxial ray intersects the mirror at $P$ and makes an angle of $-\theta$ measured from the radius from $C$. The image distance $V'F' = f$, which varies with the ray height $h$.

The distance $\overline{CF'}$ may be evaluated from $\triangle PF'C$ using the law of sines:

\[
\frac{R}{\sin[\pi - 2\theta]} = \frac{\overline{CF'}}{\sin[\theta]} \implies \overline{CF'} = R \frac{\sin[\theta]}{\sin[\pi - 2\theta]} = R \frac{\sin[\theta]}{\sin[2\theta]}
\]

\[
\implies \overline{VF'} = f = \overline{CF'} - R < 0
\]

\[
f = R \left( \frac{\sin[\theta]}{\sin[2\theta]} - 1 \right)
\]

We can evaluate this in the limit $\sin[\theta] \to \theta, \sin[2\theta] \to 2\theta$ to get the focal length of the mirror in the paraxial case:

\[
\text{paraxial case:} f \to R \left( \frac{\theta}{2\theta} - 1 \right) = -\frac{R}{2}
\]

The paraxial focal length of a spherical mirror is $f = -\frac{2}{R}$, where the negative sign corrects for the observation that a spherical with $R < 0$ makes light converge and
thus has positive power. The focal length of a spherical mirror can be put in the same form as a refracting surface by setting the second index of refraction $n_2 = -1$

$$\frac{1}{f} = -\frac{2}{R} = \frac{(n_2 - n_1)}{R} \implies n_1 = +1 \text{ and } n_2 = -1$$

$R < 0 \implies \text{concave mirror} \implies f > 0$

$R > 0 \implies \text{convex mirror} \implies f < 0$

Note that the refractive index of the medium has no effect on the power of a spherical lens because Snell’s law for reflection does not include any contribution by $n$. In other words, the reflected ray angle is not affected by the index of refraction of the medium “ahead of” the mirror.

Consider an example of a large angle, e.g., $\theta = \frac{\pi}{6}$. The focal length for this ray is:

$$f \left[ \theta = \frac{\pi}{6} \right] = R \left( \frac{\sin \left[ \frac{\pi}{6} \right]}{\sin \left[ \frac{\pi}{3} \right]} - 1 \right) \approx -0.423R$$

In words, the focal point of a spherical lens for paraxial rays is different from the focal point for nonparaxial rays, which means that rays from the same object that are collected at different ray angles do not converge to a sharp focus, thus degrading the quality of the image. This effect is spherical aberration. Probably the most famous example of an imaging mirror with spherical aberration is the Hubble Space Telescope before adding the COSTAR optical corrector system.

Rays from an object at infinity at different ray angles (or equivalently, at different ray heights at the mirror) cross the optical axis at different locations. The paraxial rays intersect the optical axis at $F'$ such that $f = -\frac{R}{2}$, but the focal point moves toward $V$ as the ray height increases.

A graph if the focal length for different ray angles is shown:
The variation in focal length $V_0 F_0 = f = CF_0 - R$ with incident ray angle $\theta$ over the interval $0 \leq \theta \leq \frac{\pi}{4}$, showing the decrease in the focal length with increasing angle. This means that the focus for rays making a larger angle is positioned closer to the vertex of the mirror.

Spherical mirrors can create a good image of an object at the center of curvature $C$, though this is not very useful because the image also is located at $C$. As the object distance increases, the spherical aberration of the mirror becomes more apparent, so that the aperture diameter of a spherical mirror typically must be small to obtain good image quality. If the object is located at $\infty$, a spherical mirror cannot give good quality unless the aperture size is quite small compared to the radius of curvature. Mathematically, it is easy to show that the appropriate mirror shape for imaging at infinity is a paraboloid: parallel rays from an object at $\infty$ reflect from a paraboloidal surface and converge to a single image point.

10.5 Systems of Thin Lenses

The images produced by systems of thin lenses may be located by finding the “intermediate” images produced by the individual lenses, which then become the objects for the next lens in the sequence. This type of analysis also is directly applicable to the “thick” lens where the surfaces take the places of the individual thin lenses. The object is labelled by $O$ and the corresponding image by $O'$, the object- and image-space focal points are $F$ and $F'$, and the object- and image-space vertices (first and last surfaces of the system) by $V$ and $V'$. 
Imaging by a system of two thin lenses $L_1$ and $L_2$ separated by the distance $d$. The object and image distances for the first lens are $s_1$ and $s_1'$ and for the second lens are $s_2$ and $s_2'$.

In this particular case, the lenses are separated by the distance $t$ and the object distance for the second lens $s_2 = t - s_1'$. The imaging equation for the first lens determines $s_1'$:

$$\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \implies \frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1} = \frac{s_1 - f_1}{s_1 f_1}$$

$$\implies s_1' = \frac{s_1 f_1}{s_1 - f_1}$$

So the object distance to the second lens is $s_2$:

$$s_2 = t - s_1' = t - \frac{s_1 f_1}{s_1 - f_1}$$

$$= \frac{s_1 t - f_1 t - s_1 f_1}{s_1 - f_1}$$

$$= \frac{s_1 (t - f_1) - f_1 t}{s_1 - f_1}$$
Now apply the imaging equation to the second lens and substitute for $s_2$:

$$\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \Rightarrow \frac{1}{s_2'} = \frac{1}{f_2} - \frac{1}{s_2}$$

$$= \frac{1}{f_2} - \frac{s_1 - f_1}{s_1 - f_1}$$

$$= \frac{s_1 (t - f_1) - f_1 t - f_2 (s_1 - f_1)}{f_2 (s_1 - f_1) - f_1 t}$$

$$= \frac{t (s_1 - f_1) - s_1 (f_1 - f_2) + f_1 f_2}{f_2 ([t (s_1 - f_1)] - s_1 f_1)}$$

$$= \frac{s_1 (t - f_1) - f_1 f_2}{f_2 ([t (s_1 - f_1)] - s_1 f_1)}$$

$$s_2' = \frac{f_2 t - \frac{s_1 f_1 f_2}{s_1 - f_1}}{(t - f_2) - \frac{s_1 f_1}{s_1 - f_1}}$$

This complicated expression determines the image distance from the second lens given the focal lengths, the “thickness” (distance between the lenses), and the object distance $s_1$.

### 10.5.1 Back Focal Distance

We would like to collect the results into a single simple equation that is analogous to the imaging equation for the single thin lens. The back focal distance $BFD$ is $s_2'$ for $s_1 = +\infty$. In other words, it is the distance from the image-space vertex to the image-space focal point: $BFD = V F'$. Note that it is NOT the focal length of the system. We'll further analyze the difference between $f_{\text{eff}}$ and $BFD$ shortly.

$$BFD \equiv VF' = \lim_{s_1 \to \infty} s_2' = \lim_{s_1 \to \infty} \frac{f_2 t - \frac{s_1 f_1 f_2}{s_1 - f_1}}{(t - f_2) - \frac{s_1 f_1}{s_1 - f_1}}$$

$$= \frac{f_2 t - \left(\frac{s_1}{s_1 - f}\right) f_1 f_2}{(t - f_2) - \left(\frac{s_1}{s_1 - f}\right) f_1}$$

$$= \frac{f_2 t - f_1 f_2}{(t - f_2) - f_1}$$

$$\lim_{s_1 \to \infty} s_2' = \frac{f_2 (t - f_1)}{(t - f_1 + f_2)} \equiv BFD$$

Note that if $t = f_1 + f_2$ then the $BFD$ is $+\infty$, so the object and image are both an infinite distance from the system. Such a system has an infinite focal length, and thus its power (reciprocal of the focal length) is zero.
10.5.2 Front Focal Distance

Similarly, the front focal distance (FFD) is $s_1$ if $s'_2 = \infty$ and is identical to $FV$. It is calculated by setting the denominator of the expression for $s'_2$ to zero:

$$(t - f_2) - \frac{s_1 f_1}{s_1 - f_1} = 0$$

$$\Rightarrow \frac{s_1 f_1}{s_1 - f_1} = t - f_2$$

$$\Rightarrow \frac{s_1}{s_1 - f_1} = \frac{t - f_2}{f_1}$$

$$\Rightarrow s_1 f_1 = (t - f_2) (s_1 - f_1)$$

$$\Rightarrow s_1 f_1 = ts_1 - tf_1 - s_1 f_2 + f_1 f_2$$

$$\Rightarrow s_1 (f_1 + f_2 - t) = f_1 f_2 - tf_1$$

$$\lim_{s'_2 \to \infty} s_1 = FV = \frac{f_1 (t - f_2)}{t - (f_1 + f_2)} = FFD$$

Note that this expression has the same form as the front focal distance except that $f_1$ and $f_2$ are “swapped;” this makes intuitive sense, because the only difference between the two cases is that the two lenses are exchanged.

10.5.3 Thin Lenses in Contact

If the two thin lenses are in contact, then $t = 0$ and the focal distances are equal to the focal length of the “equivalent single lens.” We can calculate its value by setting $t = 0$ in the equations for FFD and BFD:

$$FFD|_{t=0} = \frac{f_1 (0 - f_2)}{0 - (f_1 + f_2)} = \frac{f_1 f_2}{f_1 + f_2}$$

$$BFD|_{t=0} = \frac{f_2 (0 - f_1)}{0 - (f_1 + f_2)} = \frac{f_1 f_2}{f_1 + f_2}$$

$$f|_{t=0} = \frac{f_1 f_2}{f_1 + f_2}$$

$$\Rightarrow \frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}, \text{ if } t = 0$$
10.5 SYSTEMS OF THIN LENSES

Two thin positive lenses in contact. The focal length of the system is shorter than the focal lengths of either, and may be evaluated to see that \( f_{\text{eff}} = \frac{f_1 f_2}{f_1 + f_2} \). The image-space principal point is the location of the “equivalent thin lens”.

10.5.4 “Effective Focal Length” of a System with Two Lenses

In the general system created from two positive lenses, we need to evaluate the focal length \( f \) (or, equivalently, the power \( \varphi \)) as the “thickness” \( t \) between the lenses is increased from zero. We just showed that:

\[
\lim_{t \to 0} \left( \frac{1}{f} \right) = \frac{1}{f_1} + \frac{1}{f_2} > 0
\]

where we are assuming that the two focal lengths are positive. If \( t \) is increased from 0 until the lenses are separated by the sum of the focal lengths, then an incoming ray parallel to the axis exits parallel to the axis; we have formed a “telescope.” Since the angles relative to the axis of the incoming and outgoing rays have not changed, then this system has no power (\( \varphi = 0 \)); its focal length is infinite:

\[
\lim_{t \to f_1 + f_2} \left( \frac{1}{f} \right) = 0 \implies f_{\text{eff}} = \infty
\]

Thus the focal length of the system increased (the system power decreased) as \( t \) increased from zero. This leads us to suspect that the power of the two-lens system must have a form like:

\[
\varphi = \frac{1}{f_1} + \frac{1}{f_2} - \alpha t
\]

where \( \alpha \) is some constant with units of \((\text{length})^{-2}\). Since the only parameters used in the system are \( f_1 \) and \( f_2 \), which have dimensions of length, we might hypothesize
that the power of the system is:

\[
\varphi = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} = \varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t
\]

\[
\varphi = \frac{f_1 f_2}{(f_1 + f_2) - t}
\]

The effective focal length of the system is the reciprocal of its power:

\[
f_{eff} = \varphi^{-1} = \frac{f_1 f_2}{(f_1 + f_2) - t}
\]

It can be interpreted as the focal length of the single thin lens that generates the same outgoing ray.

The power \(\varphi\) is measured in diopters \([\text{m}^{-1}]\) if all distances are measured in \(\text{m}\). This expression satisfies the two limiting cases we proposed. If the two lenses have positive power and the separation is just less than the sum of focal lengths, the effective focal length can be very large. This is also the case if if one of the two lenses has negative power (so that the numerator is negative) and the separation is just larger than the sum of the focal lengths (so that the denominator is just smaller than zero).

### 10.5.5 Positive Lenses Separated by \(t < f_1 + f_2\)

If two positive thin lenses are separated by less than the sum of the focal lengths, the image-space focal point \(F'\) is closer to the first lens than it would have been had the second lens been absent. As shown, the effective focal length of the system is \(f_{eff} < f_1\). We can apply the equation for \(f_{eff}\) to this case to see that:

\[
f_{eff} = \frac{f_1 f_2}{(f_1 + f_2) - t} > 0
\]

\[
0 < f_{eff} < f_1 + f_2
\]

As just stated, the effective focal length \(f_{eff}\) of the system determines the location of the single thin lens that is “equivalent” to the system in image space. A corresponding point exists in object space that will be located next. The equivalent thin lens has the same focal length \(f_{eff}\) as the two-lens system but is located at the point labeled by \(H'\) in the drawing. \(H'\) is the image-space principal point, and will be discussed in more detail shortly.
10.5 SYSTEMS OF THIN LENSES

A pair of positive thin lenses separated by less than the sum of the focal lengths.

Example:

Consider a specific example with \( f_1 = 100 \text{ mm} \), \( f_2 = 50 \text{ mm} \), and \( t = 75 \text{ mm} \). The focal length of the equivalent single lens is:

\[
f_{\text{eff}} = \frac{f_1 f_2}{(f_1 + f_2) - t} = \frac{(100 \text{ mm})(50 \text{ mm})}{(100 \text{ mm} + 50 \text{ mm}) - 75 \text{ mm}} = \frac{200}{3} \text{ mm} = \frac{66\frac{2}{3}}{3} \text{ mm}
\]

The image from the first lens is formed at its focal point:

\[
s'_1 = \left( \frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left( \frac{1}{100 \text{ mm}} - \frac{1}{\infty} \right)^{-1} = 100 \text{ mm}
\]

The object distance to the second lens is therefore the difference \( t - s'_1 \):

\[
s_2 = t - s'_1 = (75 - 100) \text{ mm} = -25 \text{ mm}
\]

The image of an object located at \( s_1 = \infty \) appears at \( s'_2 \):

\[
s'_2 = \left( \frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left( \frac{1}{50 \text{ mm}} - \frac{1}{-25 \text{ mm}} \right)^{-1} = \frac{50}{3} \text{ mm} = \frac{16\frac{2}{3}}{3} \text{ mm}
\]

measured from the rear vertex \( V' \) of the system. We already know that the system focal length is \( 66\frac{2}{3} \text{ mm} \), so the image-space principal point \( H' \) (the position of the equivalent thin lens) is located \( 66\frac{2}{3} \text{ mm} \) IN FRONT of the system focal point, i.e., 50 mm in front of the second lens and 25 mm behind the first lens. We can locate this point by continuing the object ray “forward” through the system and the image ray “backward” until they intersect; this is the location of the equivalent single thin lens that creates the same object point (this location is \( H' \)). The effective focal length is the distance from \( H' \) to \( F' \), the image of an object at infinity.
Object-Space Principal Point

We have already shown how to find the location of the equivalent single lens on the “output side” by extending the rays entering and exiting the system until they meet. We can locate the equivalent single lens in “object space” by “reversing” the system, as shown in the figure. The “first” lens in the system is now \( L_2 \) with \( f_2 = 50 \text{ mm} \). The “second” lens is \( L_1 \) with \( f_1 = 100 \text{ mm} \) and the separation is \( t = 75 \text{ mm} \). The resulting effective focal length remains unchanged at \( f_{\text{eff}} = \frac{200}{3} \text{ mm} = 66\frac{2}{3} \text{ mm} \). If we bring in a ray from an object at \( \infty \), the “intermediate” image formed by \( L_2 \) is located at the focal point of \( L_2 \):

\[
s_0' = \left( \frac{1}{f_2} - \frac{1}{s_1} \right)^{-1} = \left( \frac{1}{50} - \frac{1}{\infty} \right)^{-1} = 50 \text{ mm}
\]

Thus the image distance to \( L_1 \) is:

\[
s_2 = t - s_0' = 75 - 50 = +25 \text{ mm}
\]

The image of the object at \( s_1 = \infty \) produced by the entire system is located at \( s_2' \):

\[
s_2' = \left( \frac{1}{f_1} - \frac{1}{s_2} \right)^{-1} = \left( \frac{1}{100} - \frac{1}{+25} \right)^{-1} = -\frac{100}{3} = -33\frac{1}{3} \text{ mm}
\]

measured from the “second” lens \( L_1 \) (or equivalently from the second vertex). The image is “in front” of the second lens (on the object-space side) thus is virtual. The object-space principal point \( H \) is the point such that the distance \( H \bar{V} = f = 66\frac{2}{3} \text{ mm} \), so \( H \) is located \(-33\frac{1}{3} \text{ mm} \) IN FRONT of \( L_2 \).

![Diagram of two-lens imaging system with labeled distances and focal lengths](image.png)

*The principal and focal points of the two-lens imaging system in both object and image spaces.*

When we “re-reverse” the system to graph the object- and image-space principal points, \( H \) is located “behind” the lens \( L_2 \), as shown in the graphical rendering of the entire system: The object-space principal point is the location of the equivalent thin lens if the imaging system is reversed.
Two-lens system showing the object- and image-space principal points $H$ and $H'$ and focal points $F$ and $F'$.

For a system of two thin lenses in contact, the principal points coincide with the common location of the two lenses, i.e., that $V' = H' = H = V$.

We can now use these locations of the equivalent thin lens in the two spaces to locate the images by applying the thin-lens (Gaussian) imaging equation. HOWEVER, it is VERY important to realize that the distances $s$ and $s'$ are respectively measured from the object $O$ to the object-space principal point $H$ and from the image-space principal point $H'$ to the image point $O'$.

\[
s = \overline{OH} \\
\overline{s'} = \overline{H'O'}
\]

The process is demonstrated after first locating the images via a direct calculation.

**Imaging Equation for Equivalent Single Lens (“Brute Force” Calculation)**

Now consider the location and magnification of the image created by the original two-lens imaging system (with $L_1$ in front) for an object located 1000 mm in front of the system (so that $OV = 1000$ mm). We can locate the image step by step:

**Intermediate image created by $L_1$:**

\[
s'_1 = \left( \frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left( \frac{1}{100} - \frac{1}{1000} \right)^{-1} = \frac{1000}{9} \text{ mm} \approx 111.11 \text{ mm}
\]
Transverse magnification of first image:

\[
(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{\left(\frac{1000}{9}\right)\text{mm}}{1000\text{mm}} = -\frac{1}{9}
\]

Distance from first image to \( L_2 \):

\[
s_2 = t - s'_1 = 75\text{mm} - \frac{1000}{9}\text{mm} = -\frac{325}{9}\text{mm} \approx -36.11\text{mm}
\]

Distance from \( L_2 \) to final image:

\[
s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{50} - \frac{1}{\left(-\frac{325}{9}\right)}\right)^{-1} = +\frac{650}{31} \approx +20.97\text{mm}
\]

Transverse magnification of second image:

\[
(M_T)_2 = -\frac{\left(\frac{650}{31}\right)}{\left(-\frac{325}{9}\right)} = +\frac{18}{31}
\]

The transverse magnification of the final image is the product of the transverse magnifications of the images created by the two lenses:

\[
M_T = (M_T)_1 \cdot (M_T)_2 = \left(-\frac{1}{9}\right) \left(+\frac{18}{31}\right) = -\frac{2}{31} \approx -0.065
\]

The transverse magnification indicates that the image is 

minified (or demagnified)

and

inverted.

**Brute-Force Calculation for Object at \( H \)**

Now repeat the calculation for an object is located at \( H \). In this case, the object is located “inside” the system, so the distance from the object to the first lens is:

\[
s = \overline{OH} = -\overline{VH} = -100\text{mm}
\]

The object in this case is

virtual.

The “intermediate” image created by the first lens \( L_1 \) is located at:

\[
s'_1 = \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = \left(\frac{1}{100} - \frac{1}{-100}\right)^{-1} = +50\text{mm}
\]

Note that the transverse magnification of the image created by the first lens is:

\[
(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{+50\text{mm}}{-100\text{mm}} = +\frac{1}{2}
\]
The object distance to the second lens $L_2$ is:

\[ s_2 = t - s'_1 = 75 \text{ mm} - 50 \text{ mm} = +25 \text{ mm} \]

so the object for the second lens is real. The distance to the final image is:

\[ s'_2 = \frac{1}{f_2} = \left( \frac{1}{50} - \frac{1}{25} \right)^{-1} = -50 \text{ mm} \]

The image is located 50 mm “in front” of $L_2$ (again, “inside” the system) and thus the image of the object at $H$ is virtual. The diagram shows that the image coincides with the image-space principal point $H'$, which is conveniently appropriate for our labeling convention. The transverse magnification of the image created by the second lens is:

\[ (MT)_2 = \frac{-50 \text{ mm}}{+25 \text{ mm}} = +2 \]

The transverse magnification of the final image relative to the original image is the product of the two individual magnifications:

\[ MT = (MT)_1 \cdot (MT)_2 = \left( \frac{1}{2} \right) (+2) = +1 \]

In words,

An object located at $H$ creates an image at $H'$ with transverse magnification +1.

Ray trace for object located at object-space principal point $H$ showing that the image is located at the image-space principal point $H'$. 
Yet Another Example...

What if the object located so that $\overline{OV} = \frac{100}{3}$ mm. We can locate the image step by step:

**Intermediate image created by $L_1$:**

$$s'_1 = \left( \frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left( \frac{1}{100\text{ mm}} - \frac{1}{\frac{100}{3}\text{ mm}} \right)^{-1} = -50\text{ mm} \text{ (virtual)}$$

**Transverse magnification of first image:**

$$(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{(-50\text{ mm})}{+\frac{100}{3}\text{ mm}} = +\frac{3}{2}$$

**Distance from first image to $L_2$:**

$$s_2 = t - s'_1 = 75\text{ mm} - (-50\text{ mm}) = +125\text{ mm}$$

**Distance from $L_2$ to final image:**

$$s'_2 = \left( \frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left( \frac{1}{50\text{ mm}} - \frac{1}{125\text{ mm}} \right)^{-1} = +\frac{250}{3}\text{ mm}$$

**Transverse magnification of second image:**

$$(M_T)_2 = -\frac{(+\frac{250}{3}\text{ mm})}{(+125\text{ mm})} = -\frac{2}{3}$$

The transverse magnification of the final image is the product of the transverse magnifications of the images created by the two lenses:

$$M_T = (M_T)_1 \cdot (M_T)_2 = \left( +\frac{3}{2} \right) \left( -\frac{2}{3} \right) = -1$$

Since $M_T = -1$, we know that the object is located $2f$ away and so is the image. This confirms the locations of the principal points.

**Imaging Equation for Object- and Image-Space Principal Points**

We have just seen that the object- and image-space principal points are the points related by unit magnification. They also are the “reference” locations from which the system focal length is measured:

$$f_{\text{eff}} = FH = HF^T$$
(assuming that the object space and image space are the same medium, e.g., air). In exactly the same way, these are the “reference” locations from which the object and image distances are measured for a multi-element system

$$s = \overline{OH}$$
$$s' = \overline{H'O'}$$

The ray entering the system can be modeled as traveling from the object \(O\) to the object-space principal point \(H\). The resulting outgoing (image) ray travels from the image-space principal point \(H'\) to the image point \(O'\). This may seem a little “weird”, but actually makes perfect sense if we relate the measurements to the equation for a single thin lens. In that situation, focal lengths are measured from the thin lens to its focal points. In other words, the object- and image-space vertices \(V\) and \(V'\) coincide with the principal points \(H\) and \(H'\). We know that an object located at the lens \((s = 0)\) generates an image at the lens \((s' = 0)\) with magnification of +1; the heights of the object and image at the principal points are identical. In the realistic system where the object- and image-space principal points are at different locations, the image of an object located at \(H\) has an image at \(H'\) and still with unit magnification; an object located at the object-space principal point creates an image at the image-space principal point with the transverse magnification \(M_T = +1\).

**EMPHASIS:** the principal points \(H\) and \(H'\) are the locations of the object and image related by unit transverse magnification: \(M_T = +1\)

Contrast this to the situation if the object distance from \(H\) is \(2f\), so that the image distance is also \(2f_{eff}\) and the transverse magnification is \(-1\):

$$\overline{OH} = s = 2f_{eff}$$
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{eff}}$$
$$s' = \overline{H'O'} = 2f_{eff}$$
$$M_T = -\frac{2f_{eff}}{2f_{eff}} = -1$$

The principal points are “crossed” in the imaging system considered thus far, which merely means that the object-space principal point is “behind” the image-space principal point (towards the image-space side of the lens system). Any ray cast into the system from the object point \(O\) to \(H\) creates an image ray that departs from \(H'\) at the same height \((M_T = +1)\) and directed towards the image point \(O'\):
Principal points of an imaging system: The dashed ray from the object at $\mathbf{O}$ reaches the object-space principal point $\mathbf{H}$ with height $h$. The image ray (solid line) departs from the image-space principal point $\mathbf{H'}$ with the same height $h$ and goes to the image point $\mathbf{O'}$, so that the distances $\mathbf{OH} = s$ and $\mathbf{H'O'} = s'$ satisfy the imaging equation $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{\text{eff}}}$.

Location of Image from System via Principal Points

We can also solve this problem by using the equivalent single thin lens where the distances are measured from the object- and image-space principal points. We have already shown that the system (effective) focal length is:

$$f_{\text{eff}} = +\frac{200}{3} \text{ mm}$$

The object distance is measured to the object-space principal point, which is 100 mm behind $L_1$ (or $\mathbf{V}$), thus the object distance is the distance from $\mathbf{O}$ to $L_1$ plus 100 mm:

$$s = \mathbf{OV} + \mathbf{VH} = 1000 \text{ mm} + 100 \text{ mm} = 1100 \text{ mm}$$
The single-lens imaging equation may be used to find the image distance $s'$, which now is measured from the image-space principal point $H'$:

$$s' = \left( \frac{1}{f_{eff}} - \frac{1}{s} \right)^{-1} = \left( \frac{1}{\frac{200}{3}} - \frac{1}{1100} \right)^{-1} = \frac{2200}{31} = \overline{H'O'}$$

The image distance from the vertex is calculated by subtracting the distance from the image-space principal point $H'$ to the image-space vertex $V'$:

$$\overline{V'O'} = \overline{H'O'} - \overline{HV'} = \frac{2200}{31}\text{ mm} - 50\text{ mm} = \frac{650}{31} \approx +20.97\text{ mm}$$

The resulting magnification is:

$$M_T = -\frac{s'}{s} = -\frac{\frac{2200}{31}\text{ mm}}{1100\text{ mm}} = -\frac{2}{31} \approx -0.065$$

Note that both the image distance and the transverse magnification match those obtained with the step-by-step calculation performed above.

10.5.6 Cardinal Points

The object-space and image-space focal and principal points are four of the six so-called cardinal points that completely determine the paraxial properties of an imaging system. There are three pairs of locations where one of each pair is in object space and the other is in image space. The object- and image-space focal points are $F$ and $F'$, while the principal points $H$ and $H'$ are the locations on the axis in object and image space that are images of each other with unit magnification. The nodal points $N$ and $N'$ are the points in object and image space where the ray angle entering the object-space nodal point and exiting the image-space nodal point are identical. For systems where the object and image spaces are in air (most of the systems we care about), the principal and nodal points coincide.

A table of significant points on the axis of a paraxial system is given below:
### 10.5.7 Positive Thin Lenses Separated by \( t = f_1 + f_2 \)

We’ve already looked at this example, but consider it one more time. If the two lenses are separated by the sum of the focal lengths, then an object at \( \infty \) forms an image at \( \infty \); the system focal length is infinite. Since the focal points are both located at infinity, we say that the system is *afocal*; it has zero power, i.e., the rays exit the system at the same angle that they entered it. If the focal length of the first lens is longer than that of the second, the system is a *telescope*.

Two thin lenses separated by the sum of their focal lengths. An object located an infinite distance from the first lens forms an “intermediate” image at the image-space focal point \( f_1' \) of the first lens. The second lens forms an image at infinity. Both object- and image-space focal lengths of the equivalent system are infinite: \( f = f' = \infty \). The system has “no” focal points – it is *afocal*.

The focal length of this system is:

\[
\frac{1}{f_{\text{eff}}} = \frac{1}{\infty} = 0 = \left( \frac{1}{f_1} + \frac{1}{f_2} \right) - \frac{t}{f_1 f_2}
\]

\[
= \left( \frac{1}{f_1} + \frac{1}{f_2} \right) - \frac{f_1 + f_2}{f_1 f_2}
\]

\[
= \left( \frac{1}{f_1} + \frac{1}{f_2} \right) - \left( \frac{1}{f_1} + \frac{1}{f_2} \right) = 0
\]

where \( t = f_1 + f_2 \) is the separation between the two lenses.
10.5.8 Positive Thin Lenses Separated by \( t = f_1 \) or \( t = f_2 \)

We now continue the sequence of examples for two positive lenses separated by different distances. If two positive lenses are separated by the focal length of the first lens, then the focal length of the system is:

\[
 f_{\text{eff}} \left( \text{for } t = f_1 \right) = \frac{f_1 f_2}{f_1 + f_2} - f_1 = \frac{f_1 f_2}{f_2} = f_1
\]

If separated by the focal length of the second lens, the system focal length is \( f_2 \).

\[
 f_{\text{eff}} \left( \text{for } t = f_2 \right) = \frac{f_1 f_2}{f_1 + f_2} - f_2 = \frac{f_1 f_2}{f_1} = f_2
\]

For the purpose of this example, we analyze the second case because it is the basis for probably the most common application of imaging optics. The extension to the first case is trivial. Since the focal length of the system is identical to the focal length of the second lens if the lenses are separated by \( t = f_2 \), this suggests the question of the effect of the first lens on the image.

Consider a specific case with \( f_2 = 100 \text{ mm} \) and \( f_1 = 200 \text{ mm} \). If only \( L_2 \) is present and the object distance is \( s_2 = 1100 \text{ mm} \), the image distance is:

\[
 s'_2 = \left( \frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left( \frac{1}{100} - \frac{1}{1100} \right)^{-1} = 110 \text{ mm}
\]

The associated transverse magnification is:

\[
 M_T = - \frac{s'_2}{s_2} = - \frac{110 \text{ mm}}{1100 \text{ mm}} = -\frac{1}{10}
\]
Now add $L_1$ at the front focal point of $L_2$ and find the associated image. The object distance to $L_1$ is 1100 mm $-$ 100 mm $=$ 1000 mm. The lens forms an image at distance:

$$s'_1 = \left( \frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left( \frac{1}{200 \text{ mm}} - \frac{1}{1000 \text{ mm}} \right)^{-1} = +250 \text{ mm}$$

The associated transverse magnification is:

$$(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{250 \text{ mm}}{1000 \text{ mm}} = -\frac{1}{4}$$

The object distance to the second lens is:

$$s_2 = t - s'_1 = 100 \text{ mm} - 250 \text{ mm} = -150 \text{ mm}$$

and the resulting image distance is:

$$s'_2 = \left( \frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left( \frac{1}{100} - \frac{1}{-150} \right)^{-1} = +60 \text{ mm}$$

The associated transverse magnification of the image formed by the second lens is:

$$(M_T)_2 = -\frac{60}{-150} = +\frac{2}{5}$$

The magnification of the system is the product of the magnifications:

$$M_T = (M_T)_1 (M_T)_2 = \left( -\frac{1}{4} \right) \left( +\frac{2}{5} \right) = -\frac{1}{10}$$

This is the same transverse magnification that we obtained from $L_2$ alone! The magnification is not changed by the addition of lens $L_1$! However, the position of the image HAS changed (from $+110$ mm to $+60$ mm); the image is closer to $L_2$ if $L_1$ is added.

This system demonstrates the principle of eyeglass lenses, where the corrective lens is placed at the object-space focal point of the eyeglasses. The corrective action is to move the image without changing its transverse magnification.

### 10.5.9 Positive Thin Lenses Separated by $t > f_1 + f_2$

If the two positive lenses are separated by more than the sum of the focal lengths, the focal length of the resulting system is negative:

$$f_{eff} = \frac{f_1 f_2}{(f_1 + f_2) - t} < 0$$

If the object distance is $\infty$, the first lens forms an “intermediate” image at its image-space focal point, i.e., at $s'_1 = f'_1$. Since the object distance $s_2$ measured from the
second lens is larger than \( f_2 \), a “real” image is formed by the second lens at the system focal point \( F' \). If we extend the exiting ray until it intersects the incoming ray from the object at infinity, we can locate the equivalent single thin lens for the system, i.e., the image-space principal point \( H' \). In this case, this is located farther from the second lens than the focal point. The effective focal length \( f_{eff} = \frac{H H'}{F F'} < 0 \), so the system has negative power; this system created from two positive lenses is equivalent to a single thin lens with negative power.

The system composed of two thin lenses separated by \( d > f_1 + f_2 \). The image-space focal point \( F' \) of the system is beyond the second lens, but the image-space principal point \( H' \) is located even farther from \( L_2 \). The distance \( f_{eff} = f_{eff} < 0 \), so the system has negative power!

### 10.5.10 Systems of Two Positive Thin Lenses with Different Focal Lengths

\[
\begin{align*}
  f_{eff} &= H F' = F H = \frac{f_1 f_2}{(f_1 + f_2) - t} = \frac{100 \text{ mm} \cdot 25 \text{ mm}}{100 \text{ mm} + 25 \text{ mm} - t} \\
  BFD &= V F' = \frac{f_2 (f_1 - t)}{(f_1 + f_2) - t} = \frac{25 \text{ mm} \cdot (100 \text{ mm} - t)}{(100 \text{ mm} + 25 \text{ mm}) - t} \\
  H V' &= H F' - V F' = \frac{f_1 f_2}{(f_1 + f_2) - t} - \frac{f_2 (f_1 - t)}{(f_1 + f_2) - t} = \frac{f_2 t}{(f_1 + f_2) - t} \\
  FFD &= F V = \frac{f_1 (f_2 - t)}{(f_1 + f_2) - t} = \frac{100 \text{ mm} \cdot (25 \text{ mm} - t)}{(100 \text{ mm} + 25 \text{ mm}) - t} \\
  VH &= F H - F V = \frac{f_1 t}{(f_1 + f_2) - t}
\end{align*}
\]

\[ f_1 = +100 \text{ mm} \]
\[ f_2 = +25 \text{ mm} \]
10.5.11 Newtonian Form of Imaging Equation

The familiar Gaussian form of the imaging equation is:

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}
\]

An equivalent form is obtained by defining the distances \(x\) and \(x'\) measured from the focal points:

\[
s = x + f \quad \Rightarrow \quad x = s - f
\]

\[
s' = x' + f \quad \Rightarrow \quad x' = s' - f
\]

In the case of the real image \(O'\) for the real object \(O\) shown in the figure, both \(x\) and \(x'\) are positive because the distances are measured from left to right:

\[
\begin{array}{c|c|c|c}
\hline
\text{ } & \text{FFD} & \text{BFD} & \text{ } \\
\hline
0 \text{ mm} & +20 \text{ mm} & +20 \text{ mm} & +20 \text{ mm} \\
+25 \text{ mm} & 0 \text{ mm} & +18.75 \text{ mm} & +25 \text{ mm} = f_2 \\
+50 \text{ mm} & -33\frac{1}{3} \text{ mm} & +16\frac{2}{3} \text{ mm} & +33\frac{1}{3} \text{ mm} \\
+75 \text{ mm} & -100 \text{ mm} & +12.5 \text{ mm} & +50 \text{ mm} \\
+100 \text{ mm} & -300 \text{ mm} & 0 \text{ mm} & +100 \text{ mm} = f_1 \\
+125 \text{ mm} = f_1+f_2 & \infty & \infty & \infty \\
+150 \text{ mm} & +500 \text{ mm} & +50 \text{ mm} & -100 \text{ mm} \\
+175 \text{ mm} & +300 \text{ mm} & +37.5 \text{ mm} & -50 \text{ mm} \\
+200 \text{ mm} & +233\frac{1}{3} \text{ mm} & +33\frac{1}{3} \text{ mm} & -33\frac{1}{3} \text{ mm} \\
+225 \text{ mm} & +200 \text{ mm} & +31.25 \text{ mm} & -25 \text{ mm} \\
+250 \text{ mm} & +180 \text{ mm} & +30 \text{ mm} & -20 \text{ mm} \\
\hline
\end{array}
\]

The definition of the parameters \(x, x'\) in the Newtonian form of the imaging equation. For a real image, both \(x\) and \(x'\) are positive.
By simple substitution into the imaging equation, we obtain:

\[ \frac{1}{f} = \frac{1}{x + f} + \frac{1}{x' + f} \]

\[ \frac{1}{f} = \frac{(x' + f) + (x + f)}{(x + f)(x' + f)} \]

\[ f = \frac{xx' + (x + x')f + f^2}{(x + x') + 2f} \]

\[ \Rightarrow |(x + x') + 2f| \cdot f = xx' + (x + x')f + f^2 \]

\[ \Rightarrow x \cdot x' = f^2 \]

This is the *Newtonian form* of the imaging equation. The same expression applies for virtual images, but the sign of the distances must be adjusted, as shown:

![Diagram of a system of thin lenses with positive and negative lenses.](image)

The parameters \( x, x' \) of the Newtonian form for a virtual image.

### 10.5.12 Another System of Two Thin Lenses: the Telephoto Lens

Now consider a system composed of a positive lens and a negative lens separated by slightly more than the sum of the focal lengths. For example, consider \( f_1 = +100 \text{ mm} \), \( f_2 = -25 \text{ mm} \), and \( t = +80 \text{ mm} \). The focal length of the equivalent thin lens is easy to calculate:

\[ \frac{1}{f_{\text{eff}}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \]

\[ = \left( \frac{1}{100 \text{ mm}} + \frac{1}{-25 \text{ mm}} - \frac{80 \text{ mm}}{(+100 \text{ mm})(-25 \text{ mm})} \right)^{-1} \]

\[ f_{\text{eff}} = 500 \text{ mm} \gg f_1 \]
Now locate the image-space focal point and principal point. For an object located at \( \infty \), the \( BFD \) is found by substitution into the appropriate equation:

\[
BFD = \frac{f_2 (t - f_1)}{t - (f_1 + f_2)} = \frac{(-25 \text{ mm})(80 \text{ mm} - 100 \text{ mm})}{80 \text{ mm} - (100 \text{ mm} + (-25 \text{ mm}))} = +100 \text{ mm}
\]

The image of an object at \( \infty \) is located 100 mm behind the second lens, and thus 180 mm behind the first lens. The physical length of the system when imaging an object at \( \infty \) is 180 mm, which is MUCH less than the equivalent focal length \( f_{\text{eff}} = 500 \text{ mm} \). This is an example of an optical system whose physical length is much shorter than the focal length. Such a lens is useful for photographers; it is a “short” (and portable) lens system that has a “long” focal length. Note that astronomers also find such systems to be useful.

The locations of the image-space principal point is determined from the focal distance and the equivalent focal length:

\[
\mathbf{H}' \mathbf{F}' = \mathbf{H} \mathbf{V}' + \mathbf{V} \mathbf{F}'
\]

\[
500 \text{ mm} = \mathbf{H} \mathbf{V}' + 100 \text{ mm}
\]

\[
\mathbf{H} \mathbf{V}' = +400 \text{ mm}
\]

\[
\mathbf{H} \mathbf{V} = \mathbf{H} \mathbf{V}' - \mathbf{V} \mathbf{V}' = 400 \text{ mm} - 80 \text{ mm} = +320 \text{ mm}
\]

so the principal point is located 320 mm in front of the object-space vertex \( \mathbf{V} \). A sketch of the system and the image-space cardinal points is shown:

![Image-space focal and principal points of the telephoto system. The equivalent focal length of the system is \( f_{\text{eff}} = +500 \text{ mm} \), but the image-space focal point is only \( +100 \text{ mm} \) behind the rear vertex \( \mathbf{V}' \). The image-space principal point is 500 mm in front of the focal point.](image)

The object-space focal point is located by applying the expression for the “front focal distance”:

\[
FFD = \mathbf{F} \mathbf{V} = \frac{f_1 (t - f_2)}{t - (f_1 + f_2)} = \frac{(+100 \text{ mm})(80 \text{ mm} - (-25 \text{ mm}))}{80 \text{ mm} - (100 \text{ mm} + (-25 \text{ mm}))} = +2100 \text{ mm}
\]

(not to scale)
which is far in front of the object-space vertex $V$. The object-space principal point is found from:

$$\mathbf{F}H = \mathbf{F}V + \mathbf{V}H$$

$$+500\text{ mm} = +2100\text{ mm} + \mathbf{V}H$$

$$\mathbf{V}H = 500\text{ mm} - 2100\text{ mm} = -1600\text{ mm} \implies \mathbf{H}V = -\mathbf{V}H = +1600\text{ mm}$$

So the object-space principal point is very far in front of the first vertex.

---

**Locating the Image from the Telephoto lens**

We can locate the image of an object at a finite distance (say, $\mathbf{OV} = 3000\text{ mm}$) using any of the three methods: “brute-force” calculation, by applying the Gaussian imaging formula for distances measured from the principal points, and form the Newtonian imaging equation.

**Gaussian Formula “step by step”** The distance from the object to the first thin lens is $3000\text{ mm}$, so the intermediate image distance satisfies:

$$\frac{1}{s_1} + \frac{1}{s'_1} = \frac{1}{f_1}$$

$$s'_1 = \left( \frac{1}{100\text{ mm}} - \frac{1}{3000\text{ mm}} \right)^{-1} = \frac{3000}{29}\text{ mm} \approx 103.45\text{ mm}$$

The transverse magnification of the image from the first lens is:

$$(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{1}{29}$$

The object distance to the second lens is negative:

$$s_2 = t - s'_1 = 80\text{ mm} - \frac{3000}{29}\text{ mm} = -\frac{680}{29}\text{ mm} \approx -23.45\text{ mm}$$
the object is virtual. The image distance from the second lens is:

\[
\frac{1}{s_2} + \frac{1}{s'_2} = \frac{1}{f_2}
\]

\[
s'_1 = \left( -\frac{1}{25 \text{ mm}} - \left( -\frac{29}{680 \text{ mm}} \right) \right)^{-1} = +\frac{3400}{9} \text{ mm} \approx +377.8 \text{ mm}
\]

The corresponding transverse magnification is:

\[
(M_T)_2 = -\frac{s'_2}{s_2} = -\frac{+\frac{3400}{9} \text{ mm}}{-\frac{29}{680} \text{ mm}} \approx -16.1
\]

The system magnification is the product of the component transverse magnifications:

\[
M_T = (M_T)_1 \cdot (M_T)_2 = -\frac{1}{29} \cdot -16.1 = -\frac{5}{9}
\]

**Gaussian Formula from Principal Points** Now evaluate the same image using the Gaussian formula for distances measured from the principal points. The distance from the object to the object-space principal point is:

\[
s_1 = \text{OH} = \text{OV} + \text{VH} = 3000 \text{ mm} + (-1600 \text{ mm}) = +1400 \text{ mm}
\]

The image distance measured from the image-space principal point is found from the Gaussian image formula:

\[
\frac{1}{s'} = \frac{1}{f_{\text{eff}}} - \frac{1}{s} \implies s' = \frac{\text{HO'}}{\text{HY'}} = \left( \frac{1}{500 \text{ mm}} - \frac{1}{1400 \text{ mm}} \right)^{-1} = +\frac{7000}{9} \text{ mm} \approx 777.8 \text{ mm}
\]

The distance from the rear vertex to the image is found from the known value for \( \text{HV'} = +400 \text{ mm} \):

\[
\text{V'O'} = \text{HO'} - \text{HV'} = +\frac{7000}{9} \text{ mm} - 400 \text{ mm} = \frac{3400}{9} \text{ mm} \approx 377.8 \text{ mm}
\]

thus matching the distance obtained using “brute force”. The transverse magnification of the image created by the system is:

\[
M_T = -\frac{s'}{s} = -\frac{+\frac{7000}{9} \text{ mm}}{+1400 \text{ mm}} = -\frac{5}{9}
\]

**Newtonian Formula** Now repeat the calculation for the image position using the Newtonian lens formula. The distance from the object to the object-space focal point is:

\[
x = \text{OF} = \text{OV} + \text{VF} = \text{OV} - \text{VF} = 3000 \text{ mm} - 2100 \text{ mm} = 900 \text{ mm}
\]
Therefore the distance from the image-space focal point to the image is:

\[ x' = \frac{F'O'}{x} = \frac{f_{\text{eff}}^2}{x} = \frac{(500 \text{ mm})^2}{900 \text{ mm}} = \frac{2500}{9} \text{ mm} \approx 277.8 \text{ mm} \]

So the distance from the rear (image-space) vertex \( V' \) to the image is:

\[ V'O' = VF' + FO' = 100 \text{ mm} + \frac{2500}{9} \text{ mm} = \frac{3400}{9} \text{ mm} \approx 377.8 \text{ mm} \]

which again agrees with the result obtained by the other two methods.

10.6 Stops and Pupils

In any multielement optical system, the beam of light that passes through the system is shaped like a circular “spindle” with different radii at different axial locations. The diameter of a specific element limits the size of this spindle of rays that enters or exits the system. This element is the stop of the system and may be a lens or an aperture with no power (i.e., an iris diaphragm) that is placed specifically to limit the ray cone. Obviously, an imaging system composed of a single lens is also the stop of the system. In a two-element system, the stop must be one of the two lenses; which lens is determined by the relative sizes. The image of the stop seen from the input “side” of the lens is the entrance pupil, which determines the extent of the ray cone from the object that “gets into” the optical system, and thus the “brightness” of the image. The image of the stop seen from the output “side” is the exit pupil.

The locations and sizes of the pupils are determined by applying the ray-optics imaging equation to these objects. To some, the concept of finding the image of a lens may seem confusing, but it is no different from before – just think of the lens as a regular opaque object.
A three-lens imaging system where the second lens is the stop, its image seen through the first lens is the entrance pupil, and its image seen through the last lens is the exit pupil.

Consider the stops and pupils of the Galilean telescope. Which element is the stop depends on the relative sizes of the lenses. In the first case shown below, the first lens (the objective) is small enough that it acts as the stop (and thus also the entrance pupil). The image of the objective lens seen through the eyelens is the exit pupil, and is “between” the two lenses and very small. Because the exit pupil is small, so is the field of view of the Galilean telescope. In the second example, the smaller eyelens is the stop and also the exit pupil, while the image of the eyelens seen through the objective is the entrance pupil and is far behind the eyelens and relatively large.

10.6.1 Stop and Pupils of Galilean and Keplerian Telescopes

Consider the two two-lens telescope designs; the Galilean telescope has a positive-power objective and a negative-power ocular or eyelens, while the Keplerian telescope has a positive objective and a positive eyelens. Assume that the objective is identical in the two cases with \( f_1 = +100 \text{ mm} \) and \( d_1 = 30 \text{ mm} \). The focal lengths of the oculars are \( f = \pm 15 \text{ mm} \) and \( d = +15 \text{ mm} \) (these are the approximate dimensions and focal lengths of the lenses in the OSA Optics Discovery Kit). The lenses of a telescope are separated by \( f_1 + f_2 \), or 85 mm for the Galilean and 115 mm for the Keplerian. We want to locate the stops and pupils. The stop is found by tracing a ray from an object at \( \infty \) through the edge of the first element and finding the ray height at the second lens. If this ray height is small enough to pass through the second lens, then the first lens is the stop; if not, then the second lens is the stop.

Consider the Galilean telescope first. The ray height at the first lens is the “semi-diameter” of the lens: \( \frac{d}{2} = 15 \text{ mm} \). From there, the ray height would decrease to
0 mm at a distance of \( f_1 = +100 \) mm, but it encounters the negative lens at a distance \( t = +85 \) mm. The ray height at this lens is \( \frac{100 \text{ mm} - 85 \text{ mm}}{15 \text{ mm}} = 2.25 \text{ mm} \), which is much smaller than the lens semidiameter of \( \frac{d_2}{2} = 7.5 \text{ mm} \). Since the ray “bundle” is constrained by the diameter of the first lens, it is the aperture stop of the system.

The entrance pupil is the image of the stop as seen through all of the elements that come before the stop. In this example, the first lens is also the entrance pupil, so the transverse magnification of the entrance pupil is unity.

The exit pupil is the image of the stop through all elements that come afterwards, i.e., just the negative lens. The distance to the “object” is \( f_1 + f_2 = 85 \) mm, so the imaging equation is used to locate the exit pupil and determine its magnification:

\[
\frac{1}{85 \text{ mm}} + \frac{1}{s'} = \frac{1}{f_2} = \frac{1}{-15 \text{ mm}}
\]

\[s' = \left( -\frac{1}{15 \text{ mm}} - \frac{1}{85 \text{ mm}} \right)^{-1} = -\frac{51}{4} \text{ mm} = -12.75 \text{ mm}\]

\[M_T = -\frac{s'}{s} = -\frac{-12.75 \text{ mm}}{85 \text{ mm}} = 0.15\]

The exit pupil is upright, but more important, it is virtual and thus is not accessible to the eye (you can’t put your eye at the exit pupil of a Galilean telescope).

Follow the same procedure to determine the stop and locate the pupils and their magnifications for the Keplerian telescope. The ray height at the first lens for an object located at \( \infty \) is again 15 mm. The ray height decreases to 0 mm at the focal point, but then decreases still farther until encountering the ocular lens at a distance of \( f_1 + f_2 = 115 \) mm. The ray height \( h \) at this lens is determined from similar triangles:

\[
\frac{15 \text{ mm}}{-h} = \frac{100 \text{ mm}}{15 \text{ mm}} \implies h = -2.25 \text{ mm}
\]

So the first lens is the stop and entrance pupil (with unit magnification) in this case too. The distance from the stop to the second lens is \( f_1 + f_2 = 115 \) mm, so the imaging equation for locating the exit pupil and determining its magnification are:

\[
\frac{1}{115 \text{ mm}} + \frac{1}{s'} = \frac{1}{f_2} = \frac{1}{+15 \text{ mm}}
\]

\[s' = \left( +\frac{1}{15 \text{ mm}} - \frac{1}{115 \text{ mm}} \right)^{-1} = +\frac{69}{4} \text{ mm} = +17.25 \text{ mm}\]

\[M_T = -\frac{s'}{s} = -\frac{+17.25 \text{ mm}}{85 \text{ mm}} \approx 0.203\]

The exit pupil of a Keplerian telescope is a real image – we can place our eye at it.
Galilean telescope for object at \( s_1 = +\infty \): (a) The objective lens is the stop because it limits the cone of entering rays (it also is the entrance pupil). The image of the stop seen through the eyelens is the exit pupil, and is very small; (b) The eyelens is the stop and the exit pupil. The image of the eyelens seen through the objective is the entrance pupil, and is behind the eyelens because the object distance to the objective is less than a focal length.

**10.6.2 System f-Number**

The “brightness” of a recorded image is determined by the ability of the lens to gather light and by the area of the image. This section considers the “light-gathering power” of an optical system that creates a real image, i.e., one that can be placed on a sensor (sheet of film or CCD). To create a real image, the object distance \( s \) must be larger than the focal length \( f_{\text{eff}} \). The transverse magnification of the system:

\[
M_T = -\frac{h'}{h} = -\frac{s'}{s} = -\left(\frac{1}{f_{\text{eff}}} - \frac{1}{s}\right)^{-1} = \frac{f_{\text{eff}}}{s} = -\frac{f_{\text{eff}}}{s} \left(\frac{1}{1 - \frac{f_{\text{eff}}}{s}}\right)
\]

Since \( s > f \) to ensure that the image is real, then \( \frac{f}{s} < 1 \) and the second term can be expanded using the well-known series:

\[
\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots \quad \text{if} \quad |t| < 1
\]
Hence

\[ M_T = -\frac{f_{\text{eff}}}{s} \sum_{n=0}^{\infty} \left( \frac{f_{\text{eff}}}{s} \right)^n = -f_{\text{eff}} \left( \frac{1}{s} + \frac{f_{\text{eff}}}{s^2} + \frac{f_{\text{eff}}^2}{s^3} + \cdots \right) \]

Note that \( M_T < 0 \). If \( s >> f_{\text{eff}} \) (as often is the case; the object is quite far from the system), then we can truncate the series after the first term and we can say that:

\[ M_T \approx -\frac{f_{\text{eff}}}{s} \propto f_{\text{eff}} \text{ if } f_{\text{eff}} << s \]

In words, the transverse magnification is approximately proportional to the focal length for real images of objects located many focal lengths from the system. This refers to the linear dimension, so the area of the image is proportional to \((M_T)^2 \propto f^2\).

The “brightness” of a recorded image also is determined by the ability of the lens to gather light, which is determined by the area of the entrance pupil. This is (of course) proportional to the square of the diameter \( D \) of the exit pupil and thus the the square of the diameter of the aperture stop. If we combine the contributions from the exit pupil and the transverse magnification, we can see that the “flux density” of light (or the “brightness” of the image) is proportional to

\[ \text{flux density} \propto \left( \frac{D}{f_{\text{eff}}} \right)^2 \]

The ratio \( D / f_{\text{eff}} \) is sometimes called the relative aperture and its reciprocal is the f-number or f-ratio of the system:

\[ f / \# \equiv \frac{f_{\text{eff}}}{D} \]

For a fixed focal length, a system with a smaller f-number collects more light and thus produces a brighter image.

## 10.7 Ray Tracing

The imaging equation(s) become quite complicated in systems with more than a very few lenses. However, we can determine the effect of the optical system by ray tracing, where the action on two (or more) rays is determined. Raytracing may be paraxial or exact. Historically, graphical, matrix, or worksheet ray tracing were commonly used in optical design, but most ray tracing is now implemented in computer software so that exact solutions are more commonly implemented than heretofore.

### 10.7.1 Marginal and Chief Rays

Many important characteristics of an optical system are determined by two specific rays through the system. The marginal ray travels from the optic axis at the center of the object, just grazes the edge of the stop, and then travels to the center of the
image. The chief (or principal) ray travels from the edge of the object through the center of the stop to the edge of the image. An image is created wherever the marginal ray crosses the axis. The chief ray crosses the axis at the stop and the pupils.

The marginal and chief rays for a two-element imaging system where the second element is the stop. The marginal ray comes from the center of the object \( O \), grazes the edge of the stop and through the center of the image \( O' \). The chief ray travels from the edge of the object through the center of the stop to the edge of the image.

### 10.8 Paraxial Ray Tracing Equations

Consider the schematic of a two-element optical system:

Schematic of rays in ray tracing, using the marginal ray as an example. The ray height at the \( n^{th} \) element is \( y_n \), and the ray angle during transfer between elements \( n - 1 \) and \( n \) is \( u_n \). The system has two elements, represented by the pairs of principal planes.

The two elements are represented by their two principal “planes”, which are the planes of unit magnification. The refractive power of the first element changes the ray angle of the input input ray. In the example shown, the input ray angle \( u_1 = 0 \) radians, i.e., the ray is parallel to the optical axis. The height of this ray above
the axis at the object-space principal plane $H_1$ is $y_1$ units. The ray emerges from the principal plane $H'_1$ at the same height $y_1$ but with a new ray angle $u_2$. The ray “transfers” to the second element through the distance $t_2$ in the index $n_2$ and has ray height $y_2$ at principal plane $H_2$. The ray emerges from the principal plane at the same height but a new angle $u_3$.

### 10.8.1 Paraxial Refraction

Consider refraction of a paraxial ray emitted from the object $O$ at a surface with radius of curvature $R$. For a paraxial ray, the surface may be drawn as “vertical”. The height of the ray at the surface is $y$.

![Paraxial Refraction Diagram](image)

Refraction of a paraxial ray at a surface with radius of curvature $R$ between media with refractive indices $n$ and $n'$. The ray height and angle at the surface are $y$ and $u$, respectively. The angle of the ray measured at the center of curvature is $\alpha$. The height and angle immediately after refraction are $y$ and $u'$. The object and image distances are $s$ and $s'$.

From the drawing, the incoming ray angle is:

$$ u = \tan^{-1} \left[ \frac{y}{s} \right] \cong \frac{y}{s} > 0 $$

The corresponding equation for the outgoing ray is:

$$ u' = \tan^{-1} \left[ \frac{y'}{s'} \right] \cong \frac{y'}{s'} > 0 $$

and the angle of the refraction measure from the center of curvature is:

$$ \alpha = -\tan^{-1} \left[ \frac{y}{R} \right] \cong -\frac{y}{R} $$
Snell’s law tells us that:

\[ n \sin [i] = n' \sin [i'] \]
\[ n \sin [i] = n \sin [u - \alpha] \cong n [u - \alpha] \]
\[ n' \sin [i'] = n' \sin [u' - \alpha] \cong n' [u' - \alpha] \]
\[ \Rightarrow n [u - \alpha] \cong n' [u' - \alpha] \]
\[ \Rightarrow n'u' \cong nu - n\alpha + n'\alpha = nu + \alpha (n' - n) \]

\[ n'u' \cong nu + \alpha (n' - n) \]
\[ \cong nu - y \frac{(n' - n)}{R} = nu - y \varphi \]

The paraxial refraction equation in terms of the incident angle \( u \), refracted angle \( u' \), ray height \( y \), surface power \( \varphi = \frac{1}{f} \), and indices of refraction \( n \) and \( n' \) is:

\[ n'u' = nu - y \varphi \]

10.8.2 Paraxial Transfer

The paraxial transfer equation: the ray traverses the distance \( t \) in the medium with index \( n' \). The initial and final ray heights are \( y \) and \( y' \), respectively. The angle is

\[ u' = \tan^{-1} \left( \frac{y' - y}{t} \right) \implies y' = y + tu' = y + \frac{t}{n'} (n'u') \]

The transfer equation determines the ray height \( y' \) at the next surface given the initial ray height \( y \), the distance \( t \), and the angle \( u' \). From the drawing, we have:

\[ y' = y + tu' \]
\[ y' = y + \left( \frac{t}{n'} \right) (n'u') \]
where the substitution was made to put the ray angle in the same form \( n'u' \) that appeared in the refraction equation. The distance \( \frac{1}{n'} \leq t \) is called the reduced thickness.

### 10.8.3 Linearity of the Refraction and Transfer Equations

Note that both the refraction and transfer equations are linear in the height and angle, i.e., neither includes any operations involving squares or nonlinear functions (such as sine, logarithm, or tangent). Among other things, this means that they may be scaled by direct multiplication to obtain other “equivalent” rays. For example, the output angle may be scaled by scaling the input ray angle and the height by a constant factor \( \alpha \):

\[
\alpha (nu - y\varphi) = \alpha \cdot (nu) - (\alpha \cdot y) \varphi = \alpha (n'u')
\]

We will take often advantage of this linear scaling property to scale rays to to find the exact marginal and chief rays from the provisional counterparts.

### 10.8.4 Paraxial Ray Tracing

To characterize the paraxial properties of a system, two provisional rays are traced:

1. Initial ray height (at first surface) \( y = 1.0 \), initial angle \( nu = 0 \)

2. Initial ray height (at first surface) \( y' = 0.0 \), initial angle \( n\pi = 1 \)

We have already named these rays; the first is the provisional marginal ray that intersects the optical axis at the object (and thus also at every image of the object). The second ray is called the provisional chief (or principal) ray and travels from the edge of the object to the edge of the field of view through the center of the stop (and thus through the centers of the pupils, which are images of the stop).

The process of ray tracing is perhaps best introduced by example. Consider a two-element three-surface system with three surfaces. The three radii of curvature are \( R_1 = +7.8 \text{ mm} \), \( R_2 = +10 \text{ mm} \), and \( R_3 = -6 \text{ mm} \). The distance between the first two surfaces (the thickness of the first element) and between the second and third surfaces are both 3.6 mm. The refractive index between the first two surfaces is \( n_2 = 1.336 \) and between the second and third surfaces is \( n_3 = 1.413 \). The index after the last surface is \( n_4 = n_2 = 1.336 \).
The first action of the system is paraxial refraction at the first surface, which changes the ray angle but not the ray height. The new ray angle for the provisional marginal ray is:

\[(n' u')_1 = (n u)_1 - y_1 [\text{mm}] \cdot \varphi_1 [\text{mm}^{-1}]\]
\[= 0 - (1.0) (+0.043077)\]
\[= -0.043077 \text{ radian}

(note the retention of 7 decimal places; after cascading the large number calculations for a complex system, the precision of the final results will be significantly poorer).

The paraxial transfer equation for the provisional marginal ray between the first and second surface changes the height of the ray but not the angle. The height at the second surface is:

\[y'_1 = y_1 + \left( \frac{t'}{n'} \right)_1 (n' u')_1 [\text{mm}]\]
\[= 1 + \frac{3.6}{1.336} (-0.043077) = +0.883924 \text{ mm}\]

Thus the ray exits the first surface at the “reduced angle” \(n' u' \cong -0.04\) radians and arrives at the second surface at height \(y' \cong +0.88\) mm. The corresponding equations for the chief ray at the first surface are:

\[(n' u')_1 = (n u)_1 - \varphi_1\]
\[= 1 - (0.0) (+0.043077)\]
\[= 1 \text{ radian}\]
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\[
\bar{y}_1 = \bar{y}_0 + \left( \frac{t'}{n'} \right) \bar{u}_1
\]

\[= 0 + \frac{3.6}{1.336} (1) = 2.694611 \text{ mm}\]

Since the provisional chief ray went through the center of the first surface (\(\bar{y}_1 = 0\)), the ray angle \(n'\bar{u}\) did not change. The height of the chief ray at the second surface (\(\bar{y}_1 = \bar{y}_2\)) is proportional to the initial ray angle \(\bar{y}_1\).

The equations are evaluated in sequence to compute the rays through the system. These are presented in the table. Each column in the table represents a surface in the system and the “primed” quantities refer to distances and angles following the surface. In words, \(t'\) in the first row are the distances from the surface in the column to the next surface.

<table>
<thead>
<tr>
<th>(R)</th>
<th>(t')</th>
<th>(n')</th>
<th>(\varphi = -\frac{n'-n}{R})</th>
<th>(\frac{t'}{n'})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.6</td>
<td>1.336</td>
<td>(-0.043077 \text{ mm}^{-1})</td>
<td>(-0.007700 \text{ mm}^{-1})</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\bar{y} & = 0 = y + \frac{t'}{n'} (n'u') \\
0 & = (+0.756833) + \frac{t'}{n'} (-0.059596) \\
\Rightarrow \frac{t'}{n'} & = \frac{+0.756833}{-0.059596} \approx 12.699 \text{ mm} \\
\Rightarrow t' & = 12.699 \text{ mm} \cdot n' = 12.699 \cdot 1.336 \approx 16.966 \text{ mm}
\end{align*}
\]

The height and angle of the provisional chief ray at the image location are \(\bar{y} \approx 16.78 \text{ mm}\) and \(n'u' \approx 0.91 \text{ radians}\), respectively.
This system is often used as a model for the human eye if the lens is relaxed to view objects at \( \infty \). The first surface is the cornea of the eye, while the other two surfaces are the front and back of the lens. Note that the power of the cornea (0.043077 mm\(^{-1} \) = 43 diopters) is considerably larger than the powers of the lens surfaces (7.7 and 12.8 diopters, respectively); in other words, most of the refraction of the eye system occurs at the cornea.

### 10.8.5 Matrix Formulation of Paraxial Ray Tracing

The same linear paraxial ray tracing equations may be conveniently implemented as matrices acting on ray vectors for the marginal and chief rays whose components are the height and angle. The ray vectors may be defined as:

\[
\begin{bmatrix}
y \\
nu
\end{bmatrix}, \quad \begin{bmatrix}
y' \\
\nu'
\end{bmatrix}
\]

Note that there is nothing magical about the convention for the ordering of \( y \) and \( nu \); this is the convention used by Roland Shack at the Optical Sciences Center at the University of Arizona, but Willem Brouwer wrote a book on matrix methods in optics that uses the opposite order (which Hecht also uses).

These column vectors may be combined to form a ray matrix \( \mathbf{L} \), where the columns are the marginal and chief ray vectors:

\[
\mathbf{L} \equiv \begin{bmatrix}
y & y' \\
nu & \nu'
\end{bmatrix}
\]

which may be evaluated at any point in the system. The determinant of this ray matrix is the so-called Lagrange Invariant (which we will denote by the symbol \( \aleph \), because it is the closest character available to the Cyrillic character usually used). As suggested by its name, the Lagrange Invariant is unaffected either by refraction or transfer all of the way through the system.

\[
\det \mathbf{L} = y \cdot (\nu') - (nu) \cdot y' \equiv \aleph
\]

### Refraction Matrix

Given the ray vectors or the ray matrix, we can now define operators for refraction and transfer. Recall that paraxial refraction of a marginal ray and of a chief ray at a surface with power \( \varphi \) is:

- \( n'u' = nu - y\varphi \quad \text{for marginal ray} \)
- \( n'\nu' = n\nu - y\nu \varphi \quad \text{for chief ray} \)
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The refraction process for the marginal ray may be written as a matrix $\mathcal{R}$ and the output is the product with the ray vector which will have the same ray height and a different angle:

$$
\mathcal{R} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y' \\ n'u' \end{bmatrix}
$$

It is easy to see that the form of the matrix must be:

$$
\begin{bmatrix} 1 & 0 \\ -\varphi & 1 \end{bmatrix} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y \\ -y\varphi + nu \end{bmatrix} = \begin{bmatrix} y' \\ n'u' \end{bmatrix}
$$

and its determinant is unity:

$$
\det \begin{bmatrix} 1 & 0 \\ -\varphi & 1 \end{bmatrix} = (1) (1) - (-\varphi) (0) = 1
$$

Transfer Matrix

The transfer of the marginal ray from one surface to the next is $y' = y + \frac{t'}{n'} (n'u')$, which also may be written as the product of the matrix $\mathcal{T}$ with the ray vector:

$$
\mathcal{T} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y + (nu) \left( \frac{t'}{n'} \right) \\ nu \end{bmatrix} = \begin{bmatrix} 1 & \frac{t'}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y' \\ nu \end{bmatrix}
$$

so the determinant of the transfer matrix is also 1:

$$
\det \begin{bmatrix} 1 & \frac{t'}{n'} \\ 0 & 1 \end{bmatrix} = (1) (1) - (0) \left( \frac{t'}{n'} \right) = 1
$$

Note that we could operate on the ray matrix instead of individual ray vectors: this allows us to calculate both the marginal and chief rays at the same time:

$$
\mathcal{R} \mathcal{L} = \begin{bmatrix} 1 & 0 \\ -\varphi & 1 \end{bmatrix} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y \\ nu - y\varphi \\ n'u - \varphi \end{bmatrix}
$$

$$
\mathcal{T} \mathcal{L} = \begin{bmatrix} 1 & \frac{t'}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ nu \end{bmatrix} = \begin{bmatrix} y + \left( \frac{t'}{n'} \right) (nu) \\ y + (\frac{t'}{n'}) n'u \end{bmatrix}
$$
The refraction and transfer matrices may be combined in sequence to model a complete system. If we start with the marginal ray vector at the input object, the first operation is transfer to the first surface. The next is refraction by that surface, transfer to the next, and so forth until a final transfer to the output image:

\[
T_n R_n \cdots T_2 R_2 T_1 R_1 T_0 \left( \mathbf{L}_{\text{object}} \right) = \mathbf{L}_{\text{out}}
\]

If the initial ray matrix is located at the object (as usual), the marginal ray height is zero, so the ray matrix at the object and any images has the form:

\[
\begin{bmatrix}
0 & \overline{y}_{in} \\
(nu)_{in} & (n\overline{u})_{in}
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & \overline{y}_{out} \\
(nu)_{out} & (n\overline{u})_{out}
\end{bmatrix}
\]

so the system from object to image is:

\[
T_n R_n \cdots T_2 R_2 T_1 R_1 T_0 \begin{bmatrix}
0 & \overline{y}_{in} \\
(nu)_{in} & (n\overline{u})_{in}
\end{bmatrix} = \begin{bmatrix}
0 & \overline{y}_{out} \\
(nu)_{out} & (n\overline{u})_{out}
\end{bmatrix}
\]

The matrices appear to be laid out in inverse order, i.e., the last matrix first, but the transfer matrix \( T_0 \) acts on the input ray matrix, so it must appear on the right.

### Ray Matrix for Provisional Marginal and Chief Rays

The system is characterized by using provisional marginal and chief rays located at the object. The linearity of the computations ensure that the rays may be scaled subsequently to satisfy other system constraints, such as the diameter of the stop. The provisional marginal ray at the object has height \( y = 0 \) and ray angle \( nu = \), while the provisional chief ray at the object has height \( \overline{y} = 1 \) and angle \( n\overline{u} = 0 \). Thus the provisional ray matrix at the object is:

\[
\mathbf{L}_0 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

### “Vertex-to-Vertex Matrix” for System

The optical system matrix excludes the input ray matrix, the first transfer matrix, the last transfer matrix, and the output ray matrix. It is called the “vertex-to-vertex matrix” and is labeled \( \mathbf{M}_{VV} \):

\[
\mathbf{M}_{VV} = R_n \cdots T_2 R_2 T_1 = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
10.8 PARAXIAL RAY TRACING EQUATIONS

where \(A, B, C, D\) are real numbers to be determined. Since the determinant of the matrix product is the product of the determinants, we can see that

\[
\det \mathbf{M}_{V'} = 1 \implies AD - BC = 1
\]

For example, find \(\mathbf{M}_V\) for a two-lens system with powers \(\varphi_1 = (f_1)^{-1}\) and \(\varphi_2 = (f_2)^{-1}\) separated by \(t\):

\[
\mathbf{M}_{V'} = \mathbf{R}_2 \mathbf{T}_1 \mathbf{R}_1
\]

\[
= \begin{bmatrix}
1 & 0 & 1 & t \\
-\varphi_2 & 1 & 0 & 1 \\
1 - \varphi_1 t & t \\
-(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t) & 1 - \varphi_2 t
\end{bmatrix}
\]

It is easy to confirm that the determinant of this system matrix is unity.

To illustrate, consider the system of two thin lenses in the last section with \(f_1 = 100\text{ mm}, f_2 = 50\text{ mm}\), and \(t = 75\text{ mm}\), which we showed to have \(f_{\text{eff}} = +\frac{200}{3}\text{ mm} \approx 66.7\text{ mm}\). The system matrix is:

\[
\mathbf{M}_{V'} = \begin{bmatrix}
1 - \varphi_1 t & t \\
-(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t) & 1 - \varphi_2 t
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 1 & 75\text{ mm} \\
-\frac{1}{50\text{ mm}} & 1 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
-\frac{1}{100\text{ mm}} & 1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{4} & 75\text{ mm} \\
-\frac{3}{200\text{ mm}} & -\frac{1}{2}
\end{bmatrix}
\]

So that \(A\) and \(D\) are “pure” numbers, while \(B\) and \(D\) have dimensions of length and reciprocal length, respectively. From the values in the last section, we can see that \(B = t\) and \(C = -\frac{1}{f}\), which in turn demonstrates that the power of a two-lens system is:

\[
\varphi = \frac{1}{f} = \varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}
\]

The input ray matrix consists of the provisional marginal and chief ray at the object, which “passes through” the transfer matrix from object to front surface. If the object is located 1000 mm from the first surface, the ray matrix at the front vertex of the system is:

\[
\mathbf{T}_0 \begin{bmatrix}
y \\ nu
\end{bmatrix} = \begin{bmatrix}
0 \\ 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1000\text{ mm} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
0 \\ 1
\end{bmatrix} = \begin{bmatrix}
1000\text{ mm} \\
1
\end{bmatrix}
\]
The height of the provisional marginal ray at the front vertex is 1000 mm and the angle is 1 radian (which is a huge angle, but remember that all equations are linear, so the angle and ray height can be scaled to any value). The emerging provisional marginal ray is:

\[
\begin{bmatrix}
\frac{1}{4} & 75 \text{ mm} \\
-\frac{3}{200 \text{ mm}} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1000 \text{ mm} \\
1
\end{bmatrix}
= \begin{bmatrix}
325 \text{ mm} \\
-\frac{31}{2}
\end{bmatrix}
= \begin{bmatrix}
y \\
u
\end{bmatrix}
\]

In words, the marginal ray from an object 1000 mm in front of the lens emerges with height 325 mm and angle of \(-\frac{31}{2}\) radians. To find the location of the image, find the distance until the marginal ray height \(y = 0\):

\[
\nabla \mathbf{O'} = T \begin{bmatrix}
325 \text{ mm} \\
-\frac{31}{2}
\end{bmatrix}
= \begin{bmatrix}
1 & \frac{t'}{n'} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
325 \text{ mm} \\
-\frac{31}{2}
\end{bmatrix}
= \begin{bmatrix}
0 \\
-\frac{31}{2}
\end{bmatrix}
\]

\[
\Rightarrow 325 \text{ mm} + \left( \frac{31}{2} \cdot \frac{t'}{n'} \right) = 0
\]

\[
\Rightarrow \frac{t'}{1} = 325 \text{ mm} \cdot \frac{2}{31} = \frac{650}{31} \text{ mm} \approx +20.97 \text{ mm}
\]

which agrees with the result obtained earlier. We observed that the magnification of the image in this configuration is

\[
-\frac{s'}{s} = -\frac{\overline{OH}}{\overline{H'O'}} = -\frac{2}{31}
\]

so the provisional marginal ray at the image point has the form:

\[
\begin{bmatrix}
y' \\
n'u'
\end{bmatrix}
= \begin{bmatrix}
0 \\
-\frac{31}{2}
\end{bmatrix}
= \begin{bmatrix}
0 \\
\frac{1}{M_T}
\end{bmatrix}
\]

The marginal ray out of the vertex-to-vertex matrix for the object distance \(\overline{OV} = 1000\).
Matrices that Associate Conjugate Points

We can write a general surface-to-surface matrix $M_{VV'}$ in the form:

$$M_{VV'} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the four coefficients are to be determined. The matrix that relates two image planes $O$ and $O'$ may be obtained by adding transfer matrices for the appropriate distances from the object to the front vertex ($t_1 = \overline{OV}$) and from the rear vertex to the image ($t_2 = \overline{VO'}$).

$$M_{OO'} = \begin{bmatrix} 1 \\ t_2 \\ 0 \\ 1 \end{bmatrix} M_{VV'} \begin{bmatrix} 1 \\ t_1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A + t_2C & (A + t_2C)t_1 + B + t_2D \\ C & Ct_1 + D \end{bmatrix}$$

We know that the marginal ray heights at the object and image are zero, which thus sets some limits on the “conjugate-to-conjugate” matrix:

$$\begin{bmatrix} A + t_2C & (A + t_2C)t_1 + B + t_2D \\ C & Ct_1 + D \end{bmatrix} \begin{bmatrix} 0 & \overline{y}_{in} \\ (nu)_{in} & (n\overline{u})_{in} \end{bmatrix} = \begin{bmatrix} 0 & \overline{y}_{out} \\ (nu)_{out} & (n\overline{u})_{out} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A + t_2C & (A + t_2C)t_1 + B + t_2D \\ C & Ct_1 + D \end{bmatrix} = \begin{bmatrix} 0 & \overline{y}_{out} \\ (nu)_{out} & (n\overline{u})_{out} \end{bmatrix} \cdot \begin{bmatrix} 0 & \overline{y}_{in} \\ (nu)_{in} & (n\overline{u})_{in} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{\overline{y}_{out}}{\overline{y}_{in}} & 0 \\ \frac{(n\overline{u})_{out}(nu)_{in} - (nu)_{out}(n\overline{u})_{in}}{\overline{y}_{in} (nu)_{in}} & \frac{(nu)_{out}}{(nu)_{in}} \end{bmatrix}$$

The ratio $\frac{\overline{y}_{out}}{\overline{y}_{in}} \equiv M_T$, whereas the ratio $\frac{(nu)_{out}}{(nu)_{in}} = \frac{1}{M_T}$ may be recast by applying the Lagrangian invariant:

$$C = \frac{(n\overline{u})_{out}(nu)_{in} - (nu)_{out}(n\overline{u})_{in}}{\overline{y}_{in} (nu)_{in}}$$

$$= \frac{(n\overline{u})_{out} - (nu)_{out}(n\overline{u})_{in}}{\overline{y}_{in}}$$
\[ \mathcal{N} \equiv y \cdot (n \bar{u}) - (nu) \cdot \bar{y} \\
= y_{\text{out}} \cdot (n \bar{u})_{\text{out}} - (nu)_{\text{out}} \cdot \bar{y}_{\text{out}} = \frac{y_{\text{in}} \cdot (n \bar{u})_{\text{in}} - (nu)_{\text{in}} \cdot \bar{y}_{\text{in}}}{y_{\text{in}} \cdot (n \bar{u})_{\text{in}} - (nu)_{\text{in}} \cdot \bar{y}_{\text{in}}} = 1 \]

The conjugate-to-conjugate matrix includes the leading and following ray transfers:

\[
\mathcal{M}_{00'} = \begin{bmatrix}
A + t_2 C & (A + t_2 C) t_1 + B + t_2 D \\
C & C t_1 + D
\end{bmatrix} =
\]

\[
\therefore M_T = A + t_2 C = (C t_1 + D)^{-1}
\]

\[
\varphi = -C
\]

\[
0 = (A + t_2 C) t_1 + B + t_2 D
\]

We have four equations in the four unknowns \(A, B, C, D\), which may be combined to find useful systems metrics in terms of the elements in the vertex-to-vertex matrix \(\mathcal{M}_{VV'}\):

<table>
<thead>
<tr>
<th>Metric</th>
<th>(\frac{\mathbf{OV}}{n} = \frac{t_1}{n} = \frac{D - \frac{1}{C}}{A + Ct_2})</th>
<th>(\frac{\mathbf{V}'O}{n'} = \frac{t_2}{n'} = \frac{m - A}{C} = \frac{-B - At_1}{D - Ct_1})</th>
<th>(f_{\text{eff}} = -\frac{1}{C})</th>
<th>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</th>
<th>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance from object to front vertex</td>
<td>(\frac{\mathbf{VV}}{n} = \frac{t_1}{n} = \frac{D - \frac{1}{C}}{A + Ct_2})</td>
<td>(\frac{\mathbf{V}'O}{n'} = \frac{t_2}{n'} = \frac{m - A}{C} = \frac{-B - At_1}{D - Ct_1})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
<tr>
<td>distance from rear vertex to image</td>
<td>(\frac{\mathbf{V'O}}{n'} = \frac{t_2}{n'} = \frac{m - A}{C} = \frac{-B - At_1}{D - Ct_1})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
<tr>
<td>effective focal length of system</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
<tr>
<td>front focal distance</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
<td></td>
</tr>
<tr>
<td>back focal distance</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
<tr>
<td>distance from front vertex to object-space principal point</td>
<td>(\frac{\mathbf{VV}}{n} = \frac{t_1}{n} = \frac{D - \frac{1}{C}}{A + Ct_2})</td>
<td>(\frac{\mathbf{V'O}}{n'} = \frac{t_2}{n'} = \frac{m - A}{C} = \frac{-B - At_1}{D - Ct_1})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
<tr>
<td>distance from image-space principal point to rear vertex</td>
<td>(\frac{\mathbf{VV'}}{n'} = 1 - \frac{A}{C})</td>
<td>(\frac{\mathbf{V'O}}{n'} = \frac{t_2}{n'} = \frac{m - A}{C} = \frac{-B - At_1}{D - Ct_1})</td>
<td>(f_{\text{eff}} = -\frac{1}{C})</td>
<td>(\frac{\mathbf{FFD}}{n} = \frac{\frac{\mathbf{VF}}{n}}{\frac{\mathbf{VF'}}{n'}} = -\frac{D}{C})</td>
<td>(\frac{\mathbf{BFD}}{n} = \frac{\mathbf{VF'}}{n'} = -\frac{A}{C})</td>
</tr>
</tbody>
</table>

Again, consider the example of a system composed of two thin lenses with \(f_1 = +100\) mm, \(f_2 = +50\) mm, and \(t = +75\) mm:

\[
\mathcal{M}_{VV'} = \begin{bmatrix}
1 & 0 & 175 & 1 \\
-\frac{1}{50} & 1 & 0 & 1 \\
-\frac{1}{100} & 1 & 0 & 1 \\
-\frac{3}{200} & -\frac{1}{2} & 75 & 1
\end{bmatrix}
\]
Fom the table of properties of the matrix, we see that:

\[
\begin{align*}
    f_{\text{eff}} &= -\frac{1}{C} = +\frac{200}{3}\text{ mm} \\
    FFD &= -\frac{D}{C} = -\frac{100}{3}\text{ mm} \\
    BFD &= -\frac{A}{C} = +\frac{50}{3}\text{ mm} \\
    \nabla H &= \frac{D - 1}{C} = +100\text{ mm} \\
    H'V' &= \frac{A - 1}{C} = +50\text{ mm}
\end{align*}
\]

which again match the results obtained before.

The matrix that relates the object and image planes for the two-lens system presented above is:

\[
M_{OO'} = T_2 M_{VV'} T_1 = \begin{bmatrix} 1 & \frac{650}{31} & 0 \\
0 & 1 & \frac{1}{4} \\
\frac{3}{200} & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1000 \\
\frac{3}{200} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{31} & 0 \\
-\frac{3}{200} & -\frac{31}{2} \end{bmatrix}
\]

which has the form of the principal plane matrix except the diagonal elements are not both unity. However, note that they are reciprocals of each other, so that

\[
\det \begin{bmatrix} -\frac{2}{31} & 0 \\
-\frac{3}{200} & -\frac{31}{2} \end{bmatrix} = 1
\]

We had evaluated the transverse magnification in this configuration to be \(-\frac{2}{3T}\), so we note that the upper-left component of the conjugate-to-conjugate matrix is the transverse magnification. The general form of a conjugate-to-conjugate matrix is:

\[
M_{\text{conjugate}} = \begin{bmatrix} M_T & 0 \\
-\varphi & \frac{1}{M_T} \end{bmatrix}
\]

For the two-lens system that we have used as an example with the object located
1000 mm in front of the first lens, the conjugate-to-conjugate matrix is

\[
M_{OO'} = \begin{bmatrix}
1 & \frac{t}{f} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{4} & 75 \text{ mm} \\
-\frac{3}{200 \text{ mm}} & -\frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
1 & \frac{t}{n} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1000 \text{ mm} \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 + \frac{650}{31} \text{ mm} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1}{4} & 75 \text{ mm} \\
-\frac{3}{200 \text{ mm}} & -\frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
1000 \text{ mm} \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
-\frac{2}{31} & 0 \\
-\frac{3}{200 \text{ mm}} & -\frac{31}{2}
\end{bmatrix}
\]

**“Principal Point-to-Principal Point Matrix” for System**

The conjugate-to-conjugate (object-to-image) matrix that relates the principal points of unit magnification is:

\[
M_{HH} = \begin{bmatrix}
1 & 0 \\
-\varphi & 1
\end{bmatrix}
\]

where \( \varphi \) is the power of the system, which is the ability to deviate incoming rays.

**Example of “Vertex-to-Vertex” Matrix**

To illustrate, calculate this matrix for the thin-lens telephoto considered in the last section with \( f_1 = 100 \text{ mm} \), \( f_2 = -25 \text{ mm} \), and \( t = 80 \text{ mm} \). The system matrix is:

\[
M_{VV} = \begin{bmatrix}
1 - \varphi_1 t & t \\
-(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t) & 1 - \varphi_2 t
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
-\frac{1}{25 \text{ mm}} & 1
\end{bmatrix}
\begin{bmatrix}
180 \text{ mm} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\frac{1}{100 \text{ mm}} & 1
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{5} & 80 \text{ mm} \\
-\frac{1}{500 \text{ mm}} & \frac{21}{5}
\end{bmatrix}
\]

So that \( A \) and \( D \) are “pure” numbers, while \( B \) and \( D \) have dimensions of length and reciprocal length, respectively. From the values in the last section, we can see that \( B = t \) and \( C = -\frac{1}{f} \), which in turn demonstrates that the power of a two-lens system is:

\[
\varphi = \frac{1}{f} = \varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t = \frac{1}{f_1} + \frac{1}{f_2} \frac{t}{f_1 f_2}
\]

The input ray matrix consists of the provisional marginal and chief ray at the object, which “passes through” the transfer matrix from object to front surface. If the object is located 1000 mm from the first surface, the ray matrix at the front vertex
The height of the provisional marginal ray at the front vertex is 1000 units and the angle is 1 radian (which is a huge angle, but remember that all paraxial equations are linear, so the angle and ray height can be scaled to any value).

\[
\begin{bmatrix}
\frac{1}{5} & 80 \text{ mm} \\
-\frac{1}{500 \text{ mm}} & \frac{21}{5}
\end{bmatrix}
\begin{bmatrix}
1000 \text{ mm} \\
1
\end{bmatrix}
= \begin{bmatrix}
280 \text{ mm} \\
\frac{11}{5}
\end{bmatrix}
= \begin{bmatrix} y \\ nu \end{bmatrix}
\]

In words, the marginal ray from an object 1000 mm in front of the lens emerges with height 280 mm and angle of \( +\frac{11}{5} \) radians. To find the location of the image, find the distance until the marginal ray height \( y = 0 \):

\[
\nabla O' = T \begin{bmatrix}
280 \text{ mm} \\
\frac{11}{5}
\end{bmatrix} = \begin{bmatrix} 1/nu \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix}
280 \text{ mm} \\
\frac{11}{5}
\end{bmatrix} = \begin{bmatrix} 0 \\ \frac{11}{5} 
\end{bmatrix}
\]

\[\Rightarrow 280 \text{ mm} + \left( +\frac{11}{5} \frac{t'}{n'} \right) = 0 \]
\[\Rightarrow \frac{t'}{1} = 280 \text{ mm} \cdot \frac{5}{11} = -\frac{1400}{11} \text{ mm} \approx -127.3 \text{ mm} \]

The magnification of the image in this configuration is

\[M_T = -\frac{s'}{s} = -\frac{\overline{OH}}{\overline{FO'}} = -\frac{2}{31}\]

The marginal ray out of the vertex-to-vertex matrix for the object distance \( \overline{OV} = 1000 \).
Calculating the Back Focal Distance (BFD)

The image of an object located at \( \infty \) is the image-space focal point of the system. This ray enters the system with angle \( n_u = 0 \) and arbitrary height, which we can model as \( y = 1 \). The emerging ray is:

\[
\begin{bmatrix}
\frac{1}{4} & 75 \\
-\frac{3}{200} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{4} \\
-\frac{3}{200}
\end{bmatrix}
\]

The ray height is \( \frac{1}{4} \) and the angle is \( n'u' = -\frac{3}{200} \). The distance to the point where the ray height is zero is the back focal distance:

\[
BFD = \nabla F' = T \begin{bmatrix}
\frac{1}{4} \\
-\frac{3}{200}
\end{bmatrix} = \begin{bmatrix}
1 & \frac{t'}{n'} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{1}{4} \\
-\frac{3}{200}
\end{bmatrix} = \begin{bmatrix}
0 \\
-\frac{3}{200}
\end{bmatrix}
\]

\[
\Rightarrow \frac{1}{4} + \left( -\frac{3}{200} \frac{t'}{n'} \right) = 0
\]

\[
\Rightarrow \frac{t'}{n'} = \frac{1}{4} \times \frac{200}{3} = \frac{100}{6} \approx 16.7 \text{ units}
\]

Front Focal Distance (FFD): Ray Through “Reversed” System

To find the front focal distance, we can trace the “provisional” marginal ray “backwards” through the system, or trace it through the “reversed” system where the lenses are placed in the opposite order. The “reversed” system matrix is:

\[
(MVV')_{\text{reversed}} = \begin{bmatrix}
1 & 0 \\
-\frac{1}{100} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 175 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\frac{1}{50} & 1
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{2} & 75 \\
-\frac{3}{200} & \frac{1}{4}
\end{bmatrix}
\]

Note that the “diagonal” elements of the “forward” and “reversed” vertex-to-vertex matrices are “swapped”, while the “off-diagonal” elements are identical.

If the input ray height is 1 and the angle is 0, the outgoing ray from the reversed matrix is:

\[
\begin{bmatrix}
-\frac{1}{2} & 75 \\
-\frac{3}{200} & \frac{1}{4}
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{2} \\
-\frac{3}{200}
\end{bmatrix}
\]

\[
\Rightarrow FFD = \nabla F = \frac{-\frac{1}{2}}{\left( -\frac{3}{200} \right)} = +\frac{100}{3}
\]

10.8.6 Examples of System Matrices:

Galilean Telescope made of Thin Lenses

The Galilean telescope consists of an objective lens with positive power and an eyelens with negative power separated by the sum of the focal lengths. If the focal length of the objective and eyelens are \( f_1 = +200 \) and \( f_2 = -25 \) units, the separation
10.8 PARAXIAL RAY TRACING EQUATIONS

\[ t = (200 - 25) = 175 \text{ units}. \] The system matrix is:

\[
\begin{bmatrix}
1 & 0 \\
\frac{-1}{(25)} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 175 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\frac{-1}{(+200)} & 1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{8} & 175 \\
0 & 8
\end{bmatrix}
\]

Note that the system power \( \varphi = 0 \implies f_{\text{eff}} = \infty \), which means that the system is “afocal”. The ray from an object at \( \infty \) with unit height generates the outgoing ray:

\[
\begin{bmatrix}
\frac{1}{8} & 175 \\
0 & 8
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix} y' \\
n'u'
\end{bmatrix} = \begin{bmatrix}
\frac{1}{8} \\
0
\end{bmatrix}
\]

so the outgoing ray is at height \( \frac{1}{8} \) and the angle is zero. Note that the diagonal elements are positive and the determinant is 1.

The “provisional” chief ray into the system has height 0 and angle 1; the outgoing ray is:

\[
\begin{bmatrix}
\frac{1}{8} & 175 \\
0 & 8
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix} y \\
n\pi
\end{bmatrix} = \begin{bmatrix}
175 \\
8
\end{bmatrix}
\]

So the outgoing ray angle is 8 times larger.

**Keplerian Telescope made of Thin Lenses**

The Keplerian telescope with \( f_1 = +200 \) and \( f_2 = +25 \) units with separation \( t = (200 + 25) = 225 \) units. The system matrix is:

\[
\begin{bmatrix}
1 & 0 \\
\frac{-1}{(25)} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 225 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\frac{-1}{(+200)} & 1
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{8} & 225 \\
0 & -8
\end{bmatrix}
\]

The diagonal elements are negative, the determinant is 1, and the system power \( \varphi = 0 \implies f_{\text{eff}} = \infty \). The ray from an object at \( \infty \) with unit height generates the outgoing ray:

\[
\begin{bmatrix}
-\frac{1}{8} & 225 \\
0 & -8
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix} y' \\
n'u'
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{8} \\
0
\end{bmatrix}
\]

so the outgoing ray is at height \( -\frac{1}{8} \) – the image is “inverted” and the angle is zero.

The “provisional” chief ray into the system has height 0 and angle 1; the outgoing ray is:

\[
\begin{bmatrix}
-\frac{1}{8} & 225 \\
0 & -8
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix} \overline{y}' \\
n'\overline{u}'
\end{bmatrix} = \begin{bmatrix}
225 \\
-8
\end{bmatrix}
\]

So the outgoing ray angle is 8 times larger than the incoming ray but negative.
**Thick Lens**

Consider the matrix for a thick lens with:

\[
\varphi_1 = \frac{n' - n}{R_1}
\]
\[
\varphi_2 = \frac{n - n'}{R_2}
\]

The system matrix of the thick lens is:

\[
\mathbf{M}_{VV'} = \begin{bmatrix}
1 & 0 & \frac{t'}{n'} \\
-\varphi_2 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\varphi_1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1 - \varphi_1 \frac{t'}{n'} & \frac{1}{n'} t' \\
-\left(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{t'}{n'}\right) & 1 - \varphi_2 \frac{t'}{n'}
\end{bmatrix}
\]

We can immediately identify the power of the thick lens, which may be written in the form of the effective focal length:

\[
\varphi = \varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{t'}{n'}
\]
\[
\frac{1}{f_{\text{eff}}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t'}{n' f_1 f_2}
\]

Consider an example made of glass with \(n' = 1.5\) with \(R_1 = +50\) mm and \(R_2 = -100\) mm. The thickness of the lens is 10 mm. The powers of the surfaces are:

\[
\varphi_1 = \frac{n' - n}{R_1} = \frac{1.5 - 1}{50 \text{ mm}} = +\frac{1}{50 \text{ mm}}
\]
\[
\varphi_2 = \frac{n - n'}{R_2} = \frac{1 - 1.5}{-100 \text{ mm}} = +\frac{1}{200 \text{ mm}}
\]

The system matrix is:

\[
\mathbf{M}_{VV'} = \begin{bmatrix}
1 & 0 & \frac{10 \text{ mm}}{1.5} \\
-\frac{1}{200 \text{ mm}} & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0.867 & 6.67 \text{ mm} \\
-\frac{1}{41.096 \text{ mm}} & 0.967
\end{bmatrix}
\]
The determinant is 1, as required. Substitute into the table of properties to find:

\[ f_{\text{eff}} = -\frac{1}{C} \approx +41.096 \text{ mm} \]
\[ FFD = -\frac{D}{C} \approx 0.967 \cdot 41.096 \text{ mm} = +39.745 \text{ mm} \]
\[ BFD = -\frac{A}{C} \approx 0.867 \cdot 41.096 \text{ mm} = +35.635 \text{ mm} \]
\[ \nabla \mathbf{H} = \frac{D - 1}{C} \approx (0.967 - 1) \cdot (-41.096 \text{ mm}) = +1.356 \text{ mm} \]
\[ \mathbf{H} \nabla' = \frac{A - 1}{C} \approx (0.867 - 1) \cdot (-41.096 \text{ mm}) = +5.466 \text{ mm} \]