

SIMG-712-90-20042 Solution Set #4

1. Consider a planar glass plate (i.e., the faces of the plate are parallel) of index n immersed in air

(a) Show that a ray incident at angle θ to the surface will emerge from the plate at the same angle.

Snell's Law:

$$n_1 \sin [\theta_1] = n_2 \sin [\theta'_1] \implies \theta'_1 = \sin^{-1} \left[\frac{n_1}{n_2} \sin [\theta_1] \right]$$

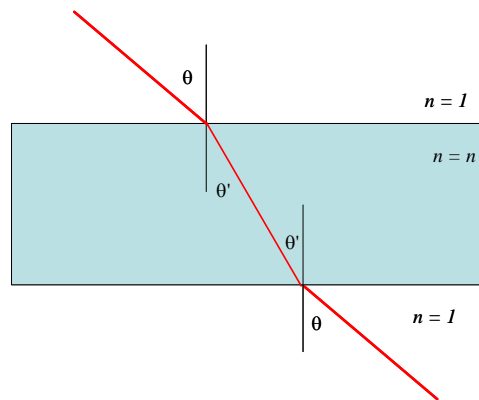
$$\rightarrow \theta'_1 = \sin^{-1} \left[\frac{1}{n_2} \sin [\theta_1] \right]$$

Since sides are parallel, $\theta_2 = \theta'_1$

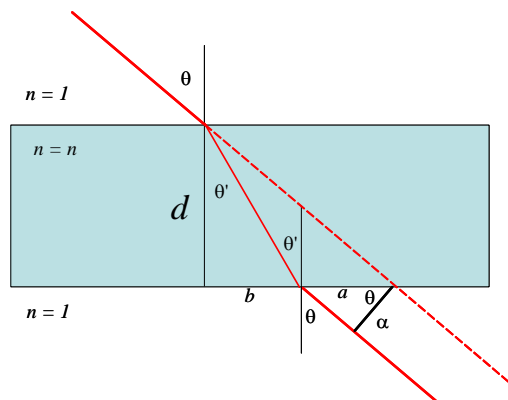
$$n_2 \sin [\theta_2] = n_1 \sin [\theta'_2]$$

$$\implies \theta'_2 = \sin^{-1} \left[\frac{n_2}{1} \sin [\theta'_1] \right] = \sin^{-1} \left[\frac{n_2}{1} \sin \left[\sin^{-1} \left[\frac{1}{n_2} \sin [\theta_1] \right] \right] \right]$$

$$= \sin^{-1} \left[\frac{n_2}{1} \frac{1}{n_2} \sin [\theta_1] \right] = \theta_1$$



(b) If the thickness of the plate is d units, derive an expression for the physical displacement a of the emerging ray relative to the original ray as a function of the incident angle.



From the drawing, we can see two reasonable distances that would be defined at a ; I have labeled them “ a ” and “ α ” – I would choose the latter (the displacement perpendicular to the path). From the drawing, we have that:

$$\begin{aligned} a + b &= d \tan [\theta] \\ b &= d \tan [\theta'] \\ \implies a &= d \tan [\theta] - d \tan [\theta'] \\ &= d \left(\frac{\sin [\theta]}{\cos [\theta]} - \frac{\sin [\theta']}{\cos [\theta']} \right) = d \left(\frac{\sin [\theta]}{\sqrt{1 - \sin^2 [\theta]}} - \frac{\sin [\theta']}{\sqrt{1 - \sin^2 [\theta']}} \right) \end{aligned}$$

Snell’s law tells us that:

$$\begin{aligned} n_1 \sin [\theta_1] &= n_2 \sin [\theta_2] \\ \implies \sin [\theta'] &= \frac{1}{n} \sin [\theta] \end{aligned}$$

So that we can evaluate a in terms of n and θ :

$$\begin{aligned} a &= d \left(\frac{\sin [\theta]}{\sqrt{1 - \sin^2 [\theta]}} - \frac{\frac{1}{n} \sin [\theta]}{\sqrt{1 - \left(\frac{1}{n}\right)^2 \sin^2 [\theta]}} \right) \\ &\boxed{a = d \sin [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{n^2 - \sin^2 [\theta]}} \right)} \end{aligned}$$

From the drawing:

$$\begin{aligned} \alpha &= a \cos [\theta] = d \sin [\theta] \cos [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{n^2 - \sin^2 [\theta]}} \right) \\ &= d \sin [\theta] \left(\frac{\cos [\theta]}{\cos [\theta]} - \frac{\cos [\theta]}{\sqrt{n^2 - \sin^2 [\theta]}} \right) \\ &\boxed{\alpha = d \sin [\theta] \left(1 - \sqrt{\frac{1 - \sin^2 [\theta]}{n^2 - \sin^2 [\theta]}} \right)} \end{aligned}$$

But it isn’t good enough to just derive the expression, we need to test it to see if it gives reasonable results. For example, if $n = 1$, then the deviation should be 0 for all θ :

$$\begin{aligned} a(n = 1) &= d \sin [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{1^2 - \sin^2 [\theta]}} \right) = 0 \\ \alpha(n = 1) &= d \sin [\theta] \left(1 - \sqrt{\frac{1 - \sin^2 [\theta]}{1 - \sin^2 [\theta]}} \right) = 0 \end{aligned}$$

So these check out. What if $n \neq 1$? In that case, if $\theta = 0$ the displacement should

still be 0:

$$a[\theta = 0] = a = d \sin[0] \left(\frac{1}{\sqrt{1 - \sin^2[0]}} - \frac{1}{\sqrt{n^2 - \sin^2[0]}} \right) = 0$$

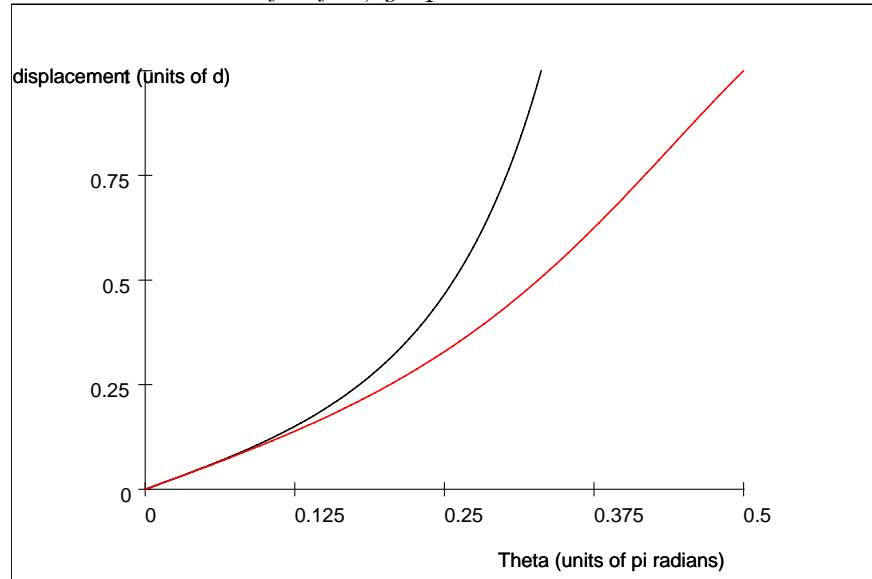
$$\alpha = d \sin[0] \left(1 - \sqrt{\frac{1 - \sin^2[0]}{n^2 - \sin^2[0]}} \right) = 0$$

So that checks out too. How about if $\theta \neq 0$ for $n \neq 0$? Say $\theta = 45^\circ$

$$a\left[\theta = \frac{\pi}{4}\right] = d \sin\left[\frac{\pi}{4}\right] \left(\frac{1}{\sqrt{1 - \sin^2\left[\frac{\pi}{4}\right]}} - \frac{1}{\sqrt{1.5^2 - \sin^2\left[\frac{\pi}{4}\right]}} \right) \cong 0.465d < \frac{d}{2}$$

$$\alpha = d \sin\left[\frac{\pi}{4}\right] \left(1 - \sqrt{\frac{1 - \sin^2\left[\frac{\pi}{4}\right]}{1.5^2 - \sin^2\left[\frac{\pi}{4}\right]}} \right) \cong 0.329d < a < \frac{d}{2}$$

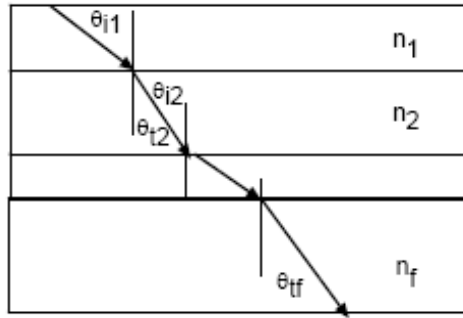
These seem to make sense. Just for fun, graph them:



Displacement a (black) and α (red) in units of the plate thickness d for $0 \leq \theta \leq \frac{\pi}{2}$.

- (c) Imagine a stratified system consisting of planar layers of transparent materials of different thicknesses. Show that the propagation direction of the emerging beam is determined by only the incident direction and the refractive indices of the initial and final layers (n_1 and n_f).

Consider the drawing:



Now apply Snell's law to each interface in sequence:

$$\begin{aligned}
 1 \sin [\theta_i] &= n_1 \sin [\theta_{i1}] \\
 n_1 \sin [\theta_{t1}] &= n_2 \sin [\theta_{i1}] \\
 n_2 \sin [\theta_{t2}] &= n_3 \sin [\theta_{i2}] \\
 &\dots \\
 n_q \sin [\theta_{tq}] &= n_f \sin [\theta_{if}]
 \end{aligned}$$

But $\theta_{t1} = \theta_{i2}$; $\theta_{t3} = \theta_{i3}$, \dots , because the sides are parallel

$$\begin{aligned}
 &\implies n_1 \sin [\theta_{t1}] = n_2 \sin [\theta_{i1}] = n_2 \sin [\theta_{t2}] = n_3 \sin [\theta_{i2}] = \\
 \dots &= n_q \sin [\theta_{tq}] = n_f \sin [\theta_{if}] \\
 &\implies n_1 \sin [\theta_{t1}] = n_f \sin [\theta_{if}] \\
 \text{If } n_1 &= n_f \implies \theta_{t1} = \theta_{if} \implies \text{rays emerge parallel}
 \end{aligned}$$

2. Three lenses with focal lengths $f_1 = +100$ mm, $f_2 = -100$ mm, and $f_3 = +100$ mm are placed in that order and each is separated from the next by $t_n = 20$ mm.

- (a) Determine the focal length of the system.

Find the equivalent single lens for the first two lenses:

$$f_{12} = \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \right)^{-1}$$

$$= \left(\frac{1}{+100 \text{ mm}} + \frac{1}{(-100 \text{ mm})} - \frac{20 \text{ mm}}{(+100 \text{ mm})(-100 \text{ mm})} \right)^{-1} = +500 \text{ mm}$$

\mathbf{F}'_{12} = image position from two lenses for object at ∞

$$\frac{1}{s'_1} = \frac{1}{f_1} \implies s'_1 = +100 \text{ mm} \implies s_2 = t_{12} - s'_1 = 20 \text{ mm} - 100 \text{ mm} = -80 \text{ mm}$$

$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-100 \text{ mm}} - \frac{1}{-80 \text{ mm}} \right)^{-1} = 400 \text{ mm}$$

Because $f_{12} = +500$ mm, \mathbf{H}_{12} is located 500 mm “in front” of \mathbf{F}'_{12} ,
or 100 mm “in front” of \mathbf{V}_2

Now combine the equivalent thin lens for 12 with 3

$$t_{23} = 100 \text{ mm} + 20 \text{ mm} = 120 \text{ mm}$$

$$f_{eff} = \left(\frac{1}{f_{12}} + \frac{1}{f_3} - \frac{t_{23}}{f_{12} f_3} \right)^{-1}$$

$$= \left(\frac{1}{+500 \text{ mm}} + \frac{1}{+100 \text{ mm}} - \frac{120 \text{ mm}}{(+500 \text{ mm})(+100 \text{ mm})} \right)^{-1} = \frac{625}{6} \text{ mm}$$

$$\boxed{f_{eff} = \frac{625}{6} \text{ mm} = 104\frac{1}{6} \text{ mm}}$$

- (b) Locate the principal and focal points.

For image-space focal point, bring in a ray from the object side parallel to the optical axis:

$$s'_1 = f_1 = +100 \text{ mm}$$

$$s_2 = t_1 - s'_1 = 20 \text{ mm} - 100 \text{ mm} = -80 \text{ mm}$$

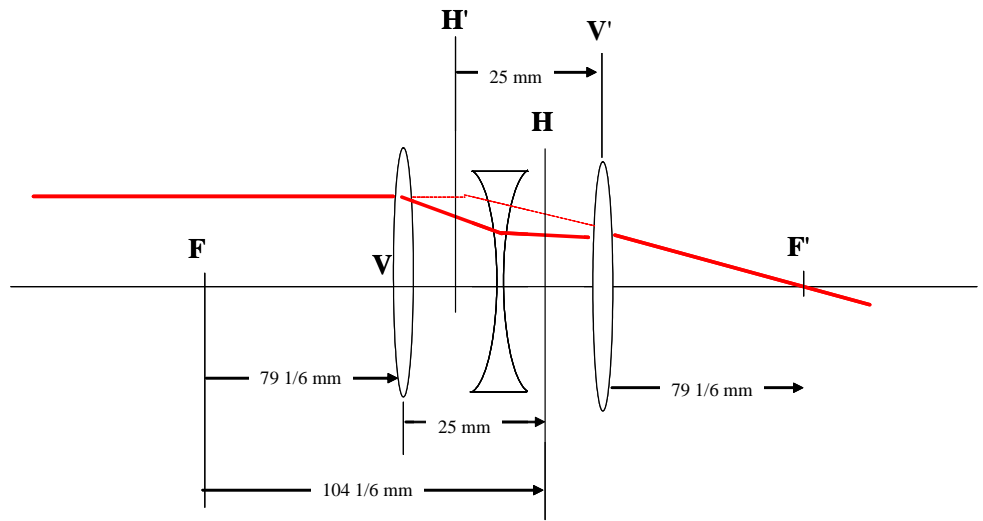
$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{(-100 \text{ mm})} - \frac{1}{(-80 \text{ mm})} \right)^{-1} = 400 \text{ mm}$$

$$s_3 = t_2 - s'_2 = 20 \text{ mm} - 400 \text{ mm} = -380 \text{ mm}$$

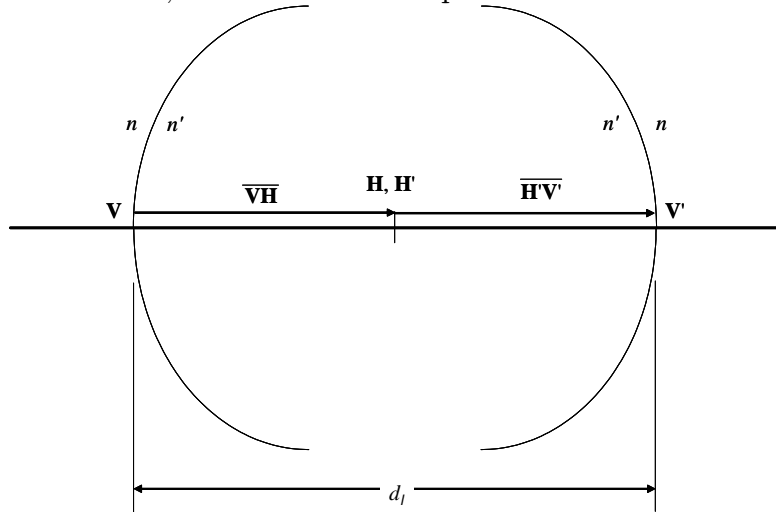
$$s'_3 = \left(\frac{1}{f_3} - \frac{1}{s_3} \right)^{-1} = \left(\frac{1}{100 \text{ mm}} - \frac{1}{-380 \text{ mm}} \right)^{-1} = \frac{475}{6} \text{ mm} = 79\frac{1}{6} \text{ mm} = \overline{\mathbf{V}'\mathbf{F}'}$$

$$\overline{\mathbf{H}'\mathbf{F}'} = f_{eff} = 104\frac{1}{6} \text{ mm} \implies \overline{\mathbf{H}'\mathbf{V}'} = \overline{\mathbf{H}'\mathbf{F}'} - \overline{\mathbf{V}'\mathbf{F}'} = 104\frac{1}{6} \text{ mm} - 79\frac{1}{6} \text{ mm} = 25 \text{ mm}$$

Because system is completely symmetric, the object -space principal and focal points are symmetrically placed.... :



3. Prove that if the principal points of a biconvex lens of thickness d in vacuum overlap midway between the vertices, then the lens is a sphere.



The principal points occur at the midpoint $\implies \overline{VH} = \overline{H'V'} = \frac{d}{2}$. I'll use the system matrix:

$$\begin{aligned} \mathcal{M}_{\mathbf{V}\mathbf{V}'} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1-n'}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n'-1}{R_1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -\varphi_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d}{n'} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\varphi_1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \varphi_1 \frac{d}{n'} & \frac{d}{n'} \\ -(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{d}{n'}) & 1 - \varphi_2 \frac{d}{n'} \end{bmatrix} \end{aligned}$$

We know that the BFD and FFD are respectively $\overline{V'F'} = -\frac{A}{C}$ and $\overline{FV} = -\frac{D}{C}$ and that these are equal:

$$\begin{aligned} BFD &= -\frac{1 - \varphi_1 \frac{d}{n'}}{-(\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{d}{n'})} = \frac{1 - \varphi_1 \frac{d}{n'}}{\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{d}{n'}} \\ &= FFD = \frac{1 - \varphi_2 \frac{d}{n'}}{\varphi_1 + \varphi_2 - \varphi_1 \varphi_2 \frac{d}{n'}} \\ BFD &= FFD \implies \varphi_1 = \varphi_2 \implies R_2 = -R_1 \\ \implies \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} 1 - \varphi \frac{d}{n'} & \frac{d}{n'} \\ -(2\varphi - \varphi^2 \frac{d}{n'}) & 1 - \varphi \frac{d}{n'} \end{bmatrix} \end{aligned}$$

Thus the radii are identical but for the sign. The distances from the vertices to the

principal points are both $\frac{d}{2}$:

$$\begin{aligned} \frac{\overline{\mathbf{V}\mathbf{H}}}{1} &= \frac{D-1}{C} = \frac{\overline{\mathbf{H}'\mathbf{V}'}}{1} = \frac{A-1}{C} = \frac{d}{2} \implies A = D = \frac{dC}{2} + 1 \\ \mathcal{M}_{\mathbf{V}\mathbf{V}'} &= \begin{bmatrix} 1 + \frac{dC}{2} & \frac{d}{n'} \\ C & 1 + \frac{dC}{2} \end{bmatrix} \\ \det[\mathcal{M}_{\mathbf{V}\mathbf{V}'}] &= 1 = \det \begin{bmatrix} 1 + \frac{dC}{2} & \frac{d}{n'} \\ C & 1 + \frac{dC}{2} \end{bmatrix} = \left(1 + \frac{dC}{2}\right)^2 - \frac{dC}{n'} \\ 1 &= Cd - C\frac{d}{n'} + \frac{1}{4}C^2d^2 + 1 \\ &\implies \frac{1}{4}C^2d^2 + Cd\left(1 - \frac{1}{n'}\right) = 0 \\ &\implies Cd = \frac{-(1 - \frac{1}{n'}) \pm \sqrt{(1 - \frac{1}{n'})^2}}{\frac{1}{2}} = \frac{-(1 - \frac{1}{n'}) \pm (1 - \frac{1}{n'})}{\frac{1}{2}} = 0, \frac{4}{n'} - 4 \end{aligned}$$

:

$Cd = 0$ is not interesting, so take

$$Cd = 4\left(\frac{1}{n'} - 1\right) \implies C = \frac{4}{d}\left(\frac{1}{n'} - 1\right)$$

But we know that

$$\begin{aligned} C &= -2\varphi + \varphi^2\frac{d}{n'} = \frac{4}{d}\left(\frac{1}{n'} - 1\right) \\ &\implies d^2(\varphi^2) + d \cdot (-2\varphi n') + 4(n' - 1) = 0 \\ d &= \frac{2\varphi n' \pm \sqrt{(-2\varphi n')^2 - 4(\varphi^2)(4(n' - 1))}}{2(\varphi^2)} \\ &= \frac{2\varphi n' \pm 2\varphi\sqrt{n'^2 - 4(n' - 1)}}{2\varphi^2} \\ d &= \frac{n' \pm \sqrt{(n')^2 - 4(n' - 1)}}{\varphi} = \frac{n' \pm \sqrt{(n')^2 - 4n' + 4}}{\varphi} = \frac{n' \pm \sqrt{(n' - 2)^2}}{\varphi} \\ d &= \frac{n' \pm (n' - 2)}{\varphi} = \frac{2}{\varphi}, \frac{2(n' - 1)}{\varphi}, \text{ the second root gives:} \\ d &= \frac{2(n' - 1)}{\left(\frac{n'-1}{R}\right)} \implies d = 2R \implies \text{sphere} \end{aligned}$$

4. The focal length of a biconvex thin lens made of glass with $n = 1.5$ is known to be 500 mm if measured in air. When immersed in a transparent liquid, the focal length is measured to be half as long. Determine the refractive index of the liquid.

$$\frac{n_1}{s_1} + \frac{n_2}{s'_1} = \frac{n_2 - n_1}{R_1}$$

$$\frac{n_2}{s_2} + \frac{n_1}{s'_2} = \frac{n_1 - n_2}{R_2}$$

$$s'_1 = -s_2 \implies \left(\frac{n_1}{s_1} + \frac{n_2}{s'_1}\right) + \left(\frac{n_2}{s_2} + \frac{n_1}{s'_2}\right) = n_1 \left(\frac{1}{s_1} + \frac{1}{s'_2}\right) = (n_2 - n_1) \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{n_1}{f_1} = (n_2 - n_1) \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \implies \boxed{\frac{1}{f} = \frac{n_2 - n_1}{n_1} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

$$\frac{1}{f_{air}} = \frac{1.5 - 1}{1} \left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{2} \left(\frac{1}{R_1} - \frac{1}{R_2}\right) = 500 \text{ mm}$$

$$\frac{1}{f_{liquid}} = \frac{1.5 - n_1}{n_1} \left(\frac{1}{R_1} - \frac{1}{R_2}\right) = 250 \text{ mm}$$

$$\frac{f_{air}}{f_{liquid}} = \frac{500 \text{ mm}}{250 \text{ mm}} = \frac{\left(\frac{1.5 - n_1}{n_1} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)}{\left(\frac{1}{2} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)} \implies \frac{1.5 - n_1}{\frac{n_1}{2}} = 2$$

$$\implies \frac{1.5 - n_1}{n_1} = 1 \implies 1.5 = 2n_1$$

$$\implies n_1 = 0.75!!!$$

(not physical, focal length in a medium has to be longer than that in air!)

5. Assume that the refractive index of a plano-convex lens is $n = 1.5$ and that the thickness is 6 mm. The radius of curvature of the convex surface is 25 mm and is positioned “forward” (towards the light from the object). Derive the system matrix (the “vertex-to-vertex matrix”) for this lens and use it to determine the focal length and to locate the principal and focal points.

The power of the front surface is:

$$\varphi = \frac{n_2 - n_1}{R_1} = \frac{1.5 - 1}{25 \text{ mm}} = +\frac{0.5}{25 \text{ mm}} = +\frac{1}{50 \text{ mm}}$$

Because the rear surface is planar, its power is 0 mm^{-1} . Using the convention in the notes, the vertex-to-vertex matrix is:

$$\begin{aligned} \mathcal{M}_{\mathbf{V}\mathbf{V}'} &= \begin{bmatrix} 1 & 0 \\ -\varphi_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{t}{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\varphi_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{6 \text{ mm}}{1.5} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{50 \text{ mm}} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.92 & 4.0 \text{ mm} \\ -\frac{1}{50 \text{ mm}} & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{aligned}$$

$$f_{eff} = -\frac{1}{C} = +50 \text{ mm}$$

$$BFD = \frac{\overline{\mathbf{V}'\mathbf{F}'}}{1} = -\frac{A}{C} = -\frac{0.92}{-\frac{1}{50 \text{ mm}}} = 46 \text{ mm}$$

$$FFD = \frac{\overline{\mathbf{F}\mathbf{V}}}{1} = -\frac{D}{C} = -\frac{1}{-\frac{1}{50 \text{ mm}}} = +50 \text{ mm}$$

$$\frac{\overline{\mathbf{H}'\mathbf{V}'}}{1} = \frac{A - 1}{C} = \frac{0.92 - 1}{-\frac{1}{50 \text{ mm}}} = 4.0 \text{ mm} \implies \mathbf{H}'\mathbf{V}' = 4.0 \text{ mm}$$

$$\frac{\overline{\mathbf{V}\mathbf{H}}}{1} = \frac{D - 1}{C} = \frac{1 - 1}{C} = 0, \text{ object-space principal point and vertex coincide.}$$

6. An object of height 20 mm is imaged by a reflective spherical ball bearing whose diameter is 25 mm. The object is 500 mm from the front vertex of the bearing. Locate the image, describe its character (real or virtual) and determine its height.

for mirror :

$$\frac{1}{s} + \frac{1}{s'} = -\frac{2}{R} \implies s' = \left(-\frac{2}{\left(\frac{25 \text{ mm}}{2}\right)} - \frac{1}{500 \text{ mm}} \right)^{-1} = -\frac{500}{81} \text{ mm} \cong -6.17 \text{ mm}$$

$$s' < 0 \implies \text{virtual image}$$

$$M_T = -\frac{s'}{s} = -\frac{\left(-\frac{500}{81} \text{ mm}\right)}{500 \text{ mm}} = +\frac{1}{81} \implies \text{upright}$$

$$\text{height } h' = M_T \cdot h = +\frac{1}{81} \cdot 20 \text{ mm} \cong 0.247 \text{ mm}$$

7. Light of a single wavelength λ_0 illuminates two small apertures in an opaque screen (the apertures can be considered to be points) located at $[x, y] = [\pm \frac{d}{2}, 0]$. The light travels down the z -axis a distance L where it encounters a screen. The pattern of irradiance on the screen is sinusoidal fringes that vary along the x -axis. Determine the period of the fringes (from maximum to maximum) as a function of L , λ_0 , and d .

From the notes,

$$\begin{aligned} D_{\text{mod}} &= \frac{\lambda}{\sin[\theta]} \\ \text{but } \sin[\theta] &\cong \frac{d}{L} \implies \boxed{D_{\text{mod}} \cong \frac{\lambda}{(\frac{d}{L})}} \\ &\implies d \cdot D_{\text{mod}} \cong L \cdot \lambda \end{aligned}$$