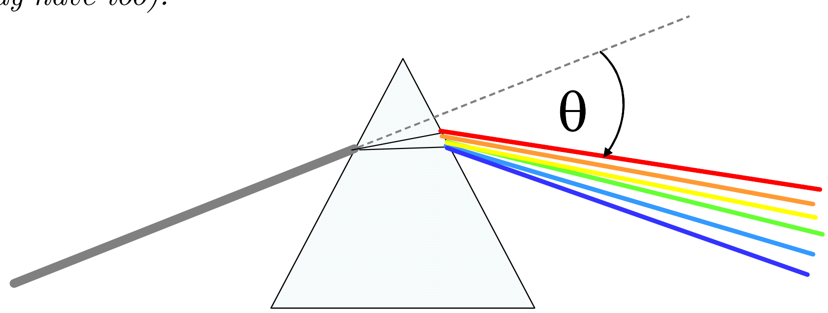
SIMG-712-90-20042 Solution Set #3

1. Fuchsin is a dye, which when in an alcohol solution appears to have a "deep red" color because it strongly absorbs green light.

   (a) If you have a hollow prism with thin glass walls that is filled with the fuchsin+alcohol solution that is used to disperse a spectrum of white light, what would you expect to observe?

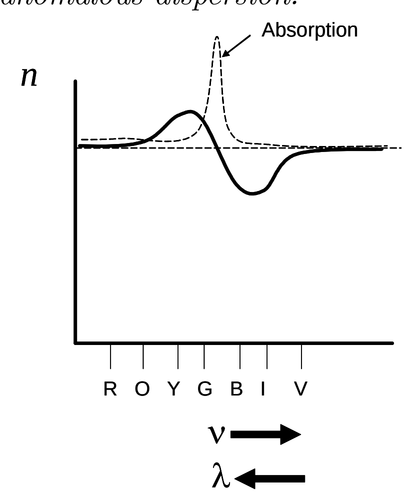
   (b) If the fuchsin dye is crystallized to make a solid, describe the spectral reflectance (reflectance that you would expect with varying wavelength) of the surface. The statement of the problem is sufficient to figure out the answer.

   (Though we've never mentioned it in this class), the normal order of dispersed colors out of a prism is the famous "ROY G. BIV" (which I learned before age 11, and you may have too).



Dispersion of a prism due to the differential index of refraction of the glass. The refractive index in normal dispersion obeys the relation: $n\,[red] < n\,[green] < n\,[blue]$.

Because fuchsin absorbs green light, the refractive indices for yellow and blue light on either side will result from anomalous dispersion.



Model of dispersion from fuchsin, showing that $n\,[yellow] > n\,[blue]$

From the sketch, we observe that the expected refractive indices follow this relation:

$$n\,[yellow] > n\,[orange] > n\,[red] > n\,[violet] > n\,[blue]$$

The spectrum observed from a fuchsin prism would be blue, violet, "green" (absorbed, so actually "black"), red, orange, yellow.

2. The refractive index of crystalline quartz (SiO_2) is measured to be 1.557 at $\lambda = 410.0\,\mathrm{nm}$ and 1.547 at $\lambda = 550.0\,\mathrm{nm}$. These can be used to calculate the first two terms in Cauchy's formula for the refractive index:

$$[\lambda] - 1 \cong A\left(1 + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \cdots\right)$$

Evaluate these terms and use them to calculate the index of refraction of quartz at $\lambda = 610.0\,\mathrm{nm}$.

$$\begin{aligned}
\lambda &= 410\,\mathrm{nm} \implies 1.557 - 1 = 0.557 = A\left(1 + \frac{B}{410^2} + \frac{C}{410^4} + \cdots\right) \\
\lambda &= 550\,\mathrm{nm} \implies 1.547 - 1 = 0.547 = A\left(1 + \frac{B}{550^2} + \frac{C}{550^4} + \cdots\right)
\end{aligned}$$

Two equations $\implies$ *can recover only two values A,B*

$$\begin{aligned}
0.557 &= A + \frac{AB}{410^2} \\
0.547 &= A + \frac{AB}{550^2}
\end{aligned}$$

To simplify the calculation, I'm going to define $AB \equiv F$*, and solve using matrices:*

$$\begin{aligned}
\begin{bmatrix} 1 & \frac{1}{410^2} \\ 1 & \frac{1}{550^2} \end{bmatrix} \begin{bmatrix} A \\ F \end{bmatrix} &= \begin{bmatrix} 0.557 \\ 0.547 \end{bmatrix} \\
&\implies \begin{bmatrix} A \\ F \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{410^2} \\ 1 & \frac{1}{550^2} \end{bmatrix}^{-1} \begin{bmatrix} 0.557 \\ 0.547 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 1 & \frac{1}{410^2} \\ 1 & \frac{1}{550^2} \end{bmatrix}^{-1} \begin{bmatrix} 0.557 \\ 0.547 \end{bmatrix} &= \begin{bmatrix} A \\ F \end{bmatrix} = \begin{bmatrix} 0.534\,49 \\ 3783.\,5 \end{bmatrix} \\
&\implies B = \frac{F}{A} = \frac{3783.5}{0.534\,49} = 7078.7
\end{aligned}$$

$$\begin{aligned}
A &\cong 0.5345 \quad \textit{(dimensionless)} \\
B &\cong 7078.7\,\mathrm{nm}^2
\end{aligned}$$

$$\implies \boxed{n\,[610\,\mathrm{nm}] \cong 1 + 0.5345\left(1 + \frac{7078.7\,\mathrm{nm}^2}{(610\,\mathrm{nm})^2}\right) \cong 1.545}$$

$$n\,[610\,\mathrm{nm}] < n\,[550\,\mathrm{nm}] < n\,[410\,\mathrm{nm}]\,,\ \textit{as expected}$$

3. The resonant frequency ω_0 of lead glass is in the ultraviolet region of the spectrum fairly near visible light, while that for fused silica is far into the ultraviolet region. Use the dispersion equation for a single resonance to sketch the approximate index of refraction for both materials in the visible spectrum. Sketch BOTH $n[\omega]$ and $n[\lambda]$ for both materials.

For frequencies ω in visible light:

$$\omega_0 [\text{Pb}] < \omega_0 [\text{Si O}_2]$$

For frequencies ω in visible light:

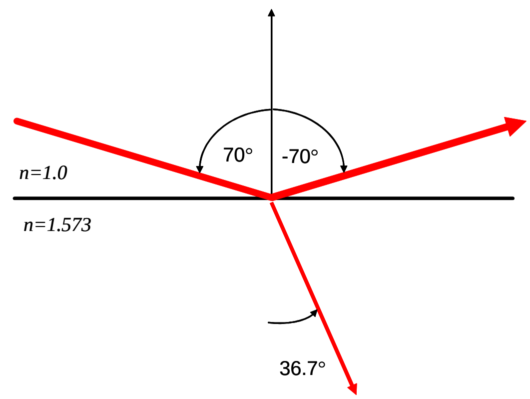
$$\begin{aligned}
(\omega_0 [\text{Pb}])^2 - \omega^2 &< (\omega_0 [\text{Si O}_2])^2 - \omega^2 \\
&\Longrightarrow \frac{1}{(\omega_0 [\text{Pb}])^2 - \omega^2} > \frac{1}{(\omega_0 [\text{Si O}_2])^2 - \omega^2} \\
&\Longrightarrow \boxed{n [\text{Pb}] > n [\text{SiO}_2] \; \textit{at visible wavelengths}}
\end{aligned}$$

4. A beam of "natural" (unpolarized) light is incident on a glass surface ($n = 1.573$) at an angle of $70°$.

   (a) Sketch the incident, reflected, and transmitted (refracted) waves.

$$\begin{aligned} n_1 \sin[\theta_0] &= n_2 \sin[\theta_t] \\ 1 \cdot \sin[70°] &= 1.573 \cdot \sin[\theta_t] \end{aligned}$$

$$\boxed{\theta_t = \sin^{-1}\left[\frac{n_1}{n_2}\sin[\theta_0]\right] \cong 0.64024 \; radians \cong 36.7°}$$



   (b) Determine the fraction of the incident light that is reflected at the surface (power, not amplitude).

   Natural light is unpolarized and thus "contains" a mixture of waves that are polarized along all possible azimuth angles ϕ, where $-\pi \leq \phi < +\pi$ in my convention. Each of these waves may be decomposed into components with the TE and TM polarizations. The reflectance of the surface is then decomposed into equal amounts of the two incident amplitudes:

$$\begin{aligned} R &= \frac{I_r}{I_0} \\ & \frac{1}{2} \; \textit{of the power is each} \; |E_{TM}|^2 \; \textit{and} \; |E_{TE}|^2 \\ \Longrightarrow & \quad R = \frac{1}{2}\frac{E_r^2}{(E_0)_{TE}^2} + \frac{1}{2}\frac{E_r^2}{(E_0)_{TM}^2} \end{aligned}$$

$$\begin{aligned} r_{TE} &= \frac{n_1 \cos[\theta_0] - n_2 \cos[\theta_t]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]} \cong -0.567 \\ R_{TE} &= \left(\frac{n_1 \cos[\theta_0] - n_2 \cos[\theta_t]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}\right)^2 \cong 0.322 \end{aligned}$$

$$\begin{aligned} r_{TM} &= \frac{+n_2 \cos[\theta_0] - n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]} \cong -0.188 \\ R_{TM} &= \left(\frac{+n_2 \cos[\theta_0] - n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}\right)^2 \cong 0.035 \end{aligned}$$

$$\frac{I_r}{I_0} = \frac{1}{2}\frac{E_r^2}{(E_0)_{TE}^2} + \frac{1}{2}\frac{E_r^2}{(E_0)_{TM}^2} = \frac{1}{2}(0.322) + \frac{1}{2}(0.035) \cong 0.178 \cong \boxed{18\% \cong \frac{I_r}{I_0}}$$

(c) In the reflected beam, determine the ratio of the component of the electric field in the plane of incidence to that at right angles to the plane of incidence.

From above:

$$\frac{E_r}{E_{TE}} \equiv -0.567$$

$$\frac{E_r}{E_{TM}} \equiv -0.188$$

$$\Longrightarrow \frac{(E_0)_{TE}}{(E_0)_{TM}} = \frac{\left(\frac{E_r}{E_{TE}}\right)}{\left(\frac{E_r}{E_{TM}}\right)} \cong \frac{-0.567}{-0.188} \cong \boxed{\frac{(E_0)_{TE}}{(E_0)_{TM}} \cong 3.02}$$

5. A fish looks upward at an unobstructed overcast sky. What is the total angular subtense of the sky viewed by the fish? Assume that the water surface is perfectly flat and that $n = 1.33$.

Find the critical angle θ_c

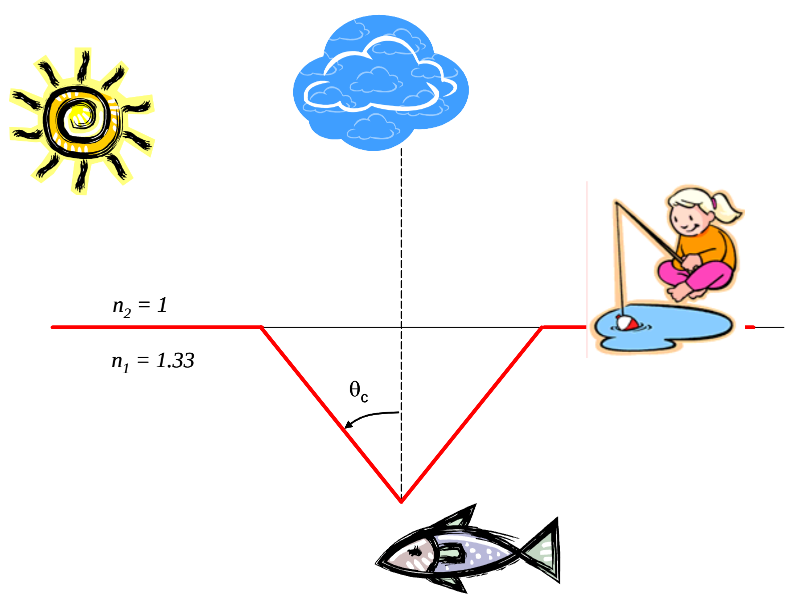
$$n_{water} \cdot \sin\theta_c = n_{air} \cdot \sin\left[\frac{\pi}{2}\right] \Longrightarrow \theta_c = \sin^{-1}\left[\frac{1}{n_{water}}\right] = \sin^{-1}\left[\frac{1}{1.33}\right]$$

$$\cong 0.851 \ \textit{radians} \cong 48.752°$$

$$\textit{total angle} = \boxed{2 \cdot \theta_c \cong 97°}$$

Fish sees 180° of angle in about 97°

Note that this is independent of the depth!



6. A diver (next to the fish) shines his flashlight at the same water surface at night so that the beam angle measured from the vertical axis is 60°.

   (a) If we assume that there is no transmitted beam when there is a reflected one, where does the flashlight beam "go?"
   From the previous problem, we know that the critical angle is $\theta_c \cong 48.75°$

$$\begin{aligned} \theta_0 &> \theta_c \Longrightarrow \textit{total internal reflection} \\ &\Longrightarrow \textit{light beam reflects back into the water} \end{aligned}$$

   (b) If oil with $n = 1.2$ is spread on the water surface. Now where does the beam of light go?
   At water-oil interface

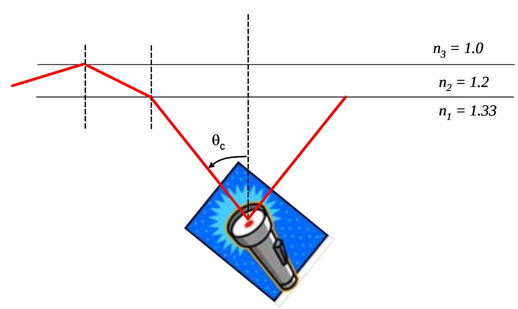
$$\theta_t = \sin^{-1}\left[\frac{n_1}{n_2}\sin[\theta_0]\right] = \sin^{-1}\left[\frac{1.33}{1.2}\sin[60°]\right] \cong 1.286 \textit{ radians} \cong 73.71°$$

   At oil-air interface:

$$\begin{aligned} \theta_t &= \sin^{-1}\left[\frac{n_2}{n_3}\sin[\theta_0']\right] \\ &= \sin^{-1}\left[\frac{1.2}{1}\sin[1.286]\right] \cong 1.5708 - 0.54402i \textit{ radians} \quad \Longrightarrow TIR \end{aligned}$$

   so the lightis still reflected at the oil-air interface

   However, if light were incident on the water-oil at the critical angle of 48.75°, *then some light would escape into the air. This is the analogue of an antireflection (AR)*
   coating



7. Total internal reflection in a certain substance occurs exactly at $\theta_0 = 45°$. Find Brewster's angle in this material.
   TIR $\Longrightarrow$ *dense-to-rare case* $\Longrightarrow$ $n_2 < n_1$

$$\begin{aligned} \theta_c &= 45° \Longrightarrow \frac{n_2}{n_1} = \sin[45°] = \frac{1}{\sqrt{2}} \\ \theta_B &\equiv \tan^{-1}\left[\frac{n_2}{n_1}\right] = \tan^{-1}\left[\frac{1}{\sqrt{2}}\right] \cong \boxed{0.615 \textit{ radians} \cong 35.26° = \theta_B} \end{aligned}$$