SIMG-712-90 Midterm Solutions

1. A lens system is composed of two thin lenses separated by a variable distance t. The prescriptions for the surfaces of the two lenses are:

$$L_1 : n = 1.5,\ R_1 = +500\,\text{mm},\ R_2 = +200\,\text{mm}$$

$$L_2 : n = 1.6,\ R_1 = -100\,\text{mm},\ R_2 = -200\,\text{mm}$$

(a) Find the focal lengths of the two thin lenses.

lensmaker's equation for lens in air:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$f_1 = \left((1.5-1)\left(\frac{1}{+500\,\text{mm}} - \frac{1}{200\,\text{mm}}\right)\right)^{-1} = \left(0.5 \cdot -\frac{3}{1000\,\text{mm}}\right)^{-1} = \frac{1}{-0.0015}\,\text{mm}$$

$$\boxed{f_1 = -666\tfrac{2}{3}\,\text{mm} = -\tfrac{2}{3}\,\text{m}}$$

$$f_2 = \left((1.6-1)\left(\frac{1}{-100\,\text{mm}} - \frac{1}{-200\,\text{mm}}\right)\right)^{-1} = \left(0.6 \cdot -\frac{1}{200\,\text{mm}}\right)^{-1}$$

$$\boxed{f_2 = -333\tfrac{1}{3}\,\text{mm} = -\tfrac{1}{3}\,\text{m}}$$

(b) Find the focal length of the system formed from these two lenses in contact.

power of a thin-lens system:

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t \implies f = \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}\right)^{-1}$$

$$t = 0 \implies f = \left(\frac{1}{-666\frac{2}{3}\,\text{mm}} + \frac{1}{-333\frac{1}{3}\,\text{mm}}\right)^{-1} = \boxed{f_{eff} = -\tfrac{2000}{9}\,\text{mm} \cong -222.22\,\text{mm}}$$

(c) Characterize the image (i.e., tell me everything you can about it) of an object created by the system of the two lenses in contact. The object is 20 mm tall and 2 mm "deep" (dimension along the direction of the optical axis). If the depth "midpoint" of the object is **O**, then the object distance is $\overline{\mathbf{OV}} = 250\,\text{mm}$.

thin-lens imaging equation : $s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{-\frac{2000}{9}\,\text{mm}} - \frac{1}{250\,\text{mm}}\right)^{-1}$

$$\boxed{s' = -\tfrac{2000}{17}\,\text{mm} \cong -117.65\,\text{mm}} \implies \textit{image is virtual}$$

$$\boxed{M_T = -\frac{-\frac{2000}{17}\,\text{mm}}{250\,\text{mm}} = +\frac{8}{17} \implies \textit{image is upright}}$$

height of image is :

$$20\,\text{mm} \cdot M_T = 20\,\text{mm} \cdot \frac{8}{17} = \frac{160}{17}\,\text{mm} \cong 9.41\,\text{mm}$$

$$M_L = -(M_T)^2 = -\left(\frac{8}{17}\right)^2 = -\frac{64}{289} = \boxed{M_L \cong -0.22}$$

"depth" of image is approximately :

$$2\,\text{mm}\cdot M_L = 2\,\text{mm}\cdot -\frac{64}{289} = -\frac{128}{289}\,\text{mm} \cong -0.44\,\text{mm}$$

(d) Find the separations t_n such that the power of the resulting lens systems are (1) $\varphi = +\frac{1}{2}$ diopters, (2) $\varphi = 0$ diopters, and (3) $\varphi = -\frac{1}{2}$ diopters.

1. *The equation for the power of a thin-lens combination is:*

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t$$

$$\varphi = \varphi = +\frac{1}{2}\,\text{m}^{-1} \implies f = 2\,\text{m} = 2000\,\text{mm}$$

$$t = \frac{\varphi - (\varphi_1 + \varphi_2)}{-\varphi_1\varphi_2}$$

$$= \frac{\frac{1}{2}\,\text{m}^{-1} - \left(-\frac{3}{2}\,\text{m}^{-1} + -3\,\text{m}^{-1}\right)}{-\left(-\frac{3}{2}\,\text{m}^{-1}\right)\left(-3\,\text{m}^{-1}\right)} = \frac{\frac{1}{2}\,\text{m}^{-1} - \left(-\frac{9}{2}\,\text{m}^{-1}\right)}{-\frac{9}{2}\,\text{m}^{-2}} = -\frac{10}{9}\,\text{m} = \boxed{t \cong -1111\,\text{mm?!?}}$$

BUT what does it mean to have a "negative separation?" You might think this implies that the first element is "behind" the second element . You might think that you can swap the lenses to make $t > 0$, BUT the focal length of a system is not changed by "swapping" the lenses:

$$\begin{aligned}
\varphi &= \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t = \varphi_2 + \varphi_1 - \varphi_2\varphi_1 t \\
&= \left(-3\,\text{m}^{-1}\right) + \left(-\frac{3}{2}\,\text{m}^{-1}\right) - \left(-3\,\text{m}^{-1}\right)\left(-\frac{3}{2}\,\text{m}^{-1}\right)\left(+\frac{10}{9}\,\text{m}\right) = -\frac{19}{2}\,\text{m}^{-1} \\
f_{eff} &= -\frac{2}{19}\,\text{m} \cong -105.3\,\text{mm}
\end{aligned}$$

Something is funny! Mull this over as we go on to the next example

2. *A system power of 0 Diopters corresponds to an infinite system focal length – this is a telescope. Try "plugging and chugging:"*

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t$$

$$\varphi = 0 \implies f = \infty \implies t = \frac{\varphi_1 + \varphi_2}{\varphi_1\varphi_2}$$

$$t = \frac{\left(-3\,\text{m}^{-1}\right) + \left(-\frac{3}{2}\,\text{m}^{-1}\right)}{\left(-3\,\text{m}^{-1}\right)\left(-\frac{3}{2}\,\text{m}^{-1}\right)} = \frac{-\frac{9}{2}\,\text{m}^{-1}}{-\frac{9}{2}\,\text{m}^{-2}} = -1\,\text{m}$$

$$\boxed{t = -1000\,\text{mm}\quad (?!!??)}$$

Again, the "negative separation" of the two lenses. The system power resulting from swapping the two lenses to make $t > 0$ is:

Lenses with negative separation

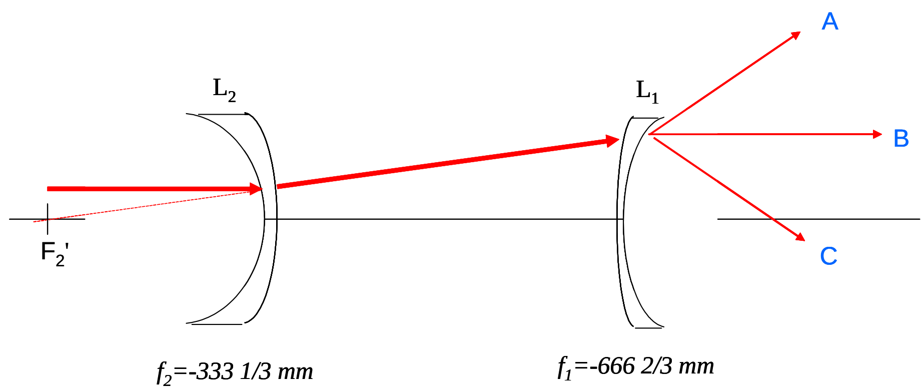
$$\begin{aligned}
\varphi &= \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t \\
&= \left(-3\,\text{m}^{-1}\right) + \left(-\frac{3}{2}\,\text{m}^{-1}\right) - \left(-3\,\text{m}^{-1}\right)\left(-\frac{3}{2}\,\text{m}^{-1}\right)(-1\,\text{m}) = 0\,\text{m}^{-1} \\
\implies &\ f_{eff} = \infty
\end{aligned}$$

$$\begin{aligned}
& \textit{Lenses with positive separation} \\
\varphi &= \varphi_2 + \varphi_1 - \varphi_2\varphi_1 t \\
&= \left(-\frac{3}{2}\,\mathrm{m}^{-1}\right) + \left(-3\,\mathrm{m}^{-1}\right) - \left(-\frac{3}{2}\,\mathrm{m}^{-1}\right)\left(-3\,\mathrm{m}^{-1}\right)(+1\,\mathrm{m}) = -9\,\mathrm{m}^{-1} \\
\Longrightarrow\ & f_{eff} \cong -111.1\,\mathrm{mm}
\end{aligned}$$

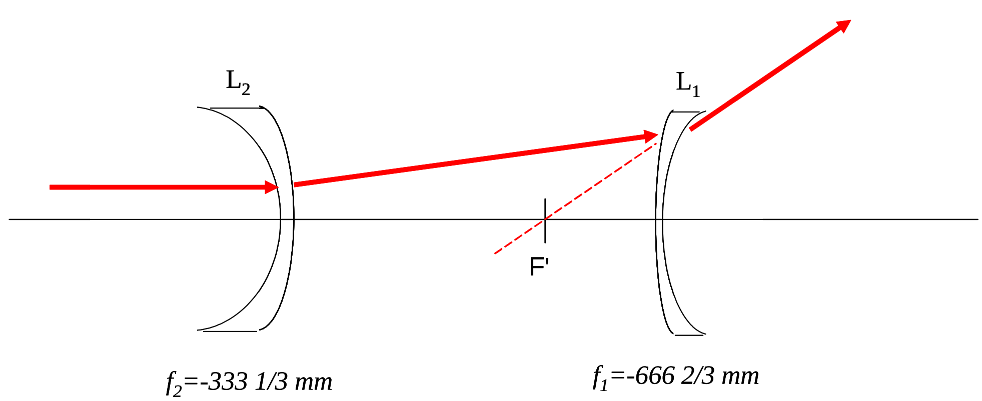
The lesson here is:

You Can't Make a Telescope from Two Negative Lenses!

If you interpreted $t = -1000\,\mathrm{mm}$ to mean that lens 2 is in front of lens 1, do the sketch:



Where does the ray out of the second lens ("L_1" since the order of the lenses was reversed)? Direction **B** *is where you want it to go if the system is to have infinite focal length, but this would require the lens to make the rays* ***converge*** *rather than* ***diverge****, as a negative lens would do. To make the ray go in the direction* ***C*** *would require that the focal length of L_1 be positive but even shorter (more powerful lens) than to get ray* ***B****. The ray must diverge from a negative, thus the only choice is direction* ***A****. Had the lenses been placed in the original order, the same situation would result. The image-space focal point of a system created from two negative lenses is virtual, the power is negative.This demonstrates that you can't make a telescope from two negative lenses.*



3. $\varphi = -\frac{1}{2}$ *diopters:*

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1 \varphi_2 t$$

$$t = \frac{\varphi - (\varphi_1 + \varphi_2)}{-\varphi_1 \varphi_2} = \frac{-\frac{1}{2}\,\mathrm{m}^{-1} - \left(-\frac{3}{2}\,\mathrm{m}^{-1} + -3\,\mathrm{m}^{-1}\right)}{-\left(-\frac{3}{2}\,\mathrm{m}^{-1}\right)\left(-3\,\mathrm{m}^{-1}\right)} = -\frac{8}{9}\,\mathrm{m} = \boxed{t \cong -889\,\mathrm{mm}!?!}$$

The same problem!
We first calculated the focal length of the two thin lenses in contact:

$$f_{eff}\,(t = 0) = -222.22\,\mathrm{mm} \implies \varphi = -\frac{9}{2}\,\mathrm{m}^{-1}$$

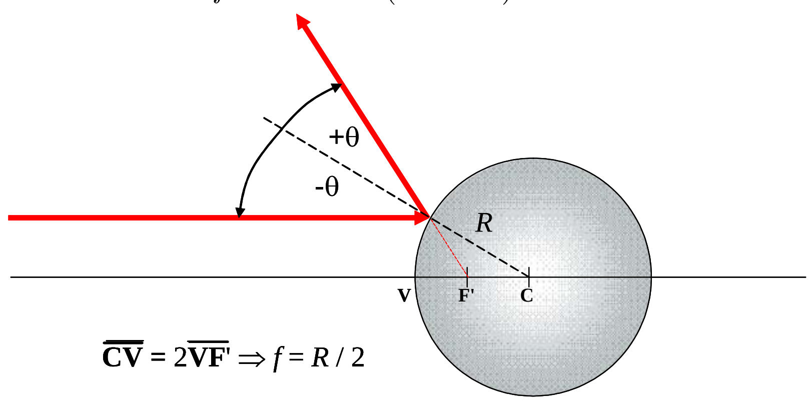
We also know that the power will get "smaller" (more negative with larger magnitude) if we increase t. This means that the system focal length will get "smaller and still negative), so the focal length becomes "shorter".

(e) Sketch the systems for each of the separations, showing the path traveled by a ray entering the system parallel to the optical axis (i.e., from an object an infinite distance away).

Done above for one case; note that none of the prescribed systems can be constructed using the lenses provided!

2. A reflective sphere (imagine a ball bearing) of diameter $d = 50\,\text{mm}$ acts as a spherical mirror that can be used to image objects.

   (a) Determine the focal length of the imaging "system" composed of this sphere.
   Draw the picture and use the reflection law $(\theta \to -\theta)$

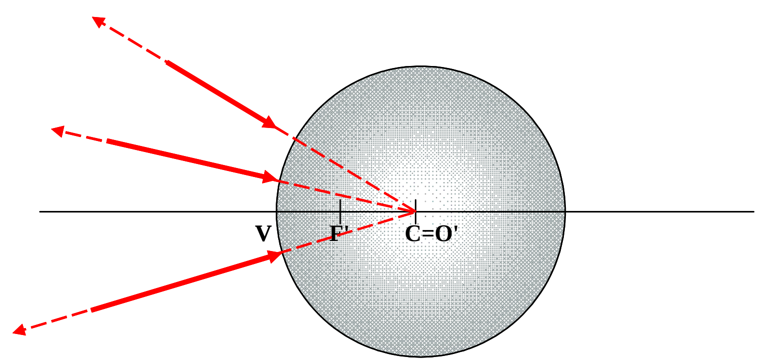


$$\begin{aligned} d &= 50\,\text{mm} \Longrightarrow R = 25\,\text{mm} \\ &\Longrightarrow \textit{paraxial focal length } \boxed{f = -\frac{R}{2} = -12.5\,\text{mm}} \end{aligned}$$

   which can be inferred from the sketch

   (b) Sketch the "system," including the location of the *image-space* focal and principal points (no need to do the object-space focal and principal points, but you may if you wish).
   The focal point is shown in the sketch in 2(a) above. What about the image-space principal point? It's the point that is one focal length away from the image-space focal point. From the drawing you can see that **H′** *coincides with* **V**. *This makes sense because it is the location of the image point if the transverse magnification is* $M_T = +1$. *By analogy with the thin lens (where the principal points coincide with* **V** *and* **V′**, *the principal points of the mirror coincide with* **V′** *(and thus with* **V** *).*

   (c) Determine the location of the input object that produces a paraxial image at the center of the sphere.
   I find it useful to draw a sketch. Since it is a mirror system, the image will be virtual. The incoming rays must be "aimed" at the center of the sphere to appear to emerge from the center of the sphere.



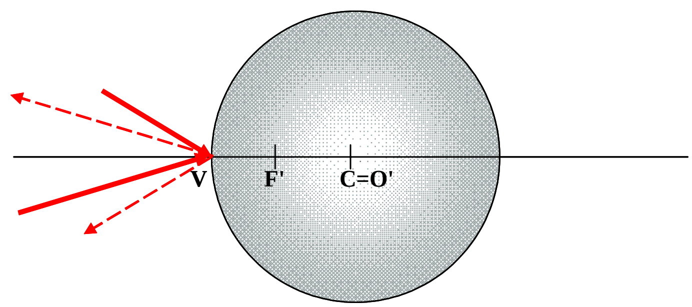
The distance can also be evaluated from the Gaussian imaging equation with $\overline{\mathbf{OV}} = -R = -25\,\text{mm}$:

$$s = \left(\frac{1}{f} - \frac{1}{s'}\right)^{-1} = \left(\frac{1}{-12.5\,\text{mm}} - \frac{1}{-25\,\text{mm}}\right)^{-1} = \boxed{-25.0\,\text{mm} = s}$$

which also is at the center of the sphere!

(d) Determine the location of the input object that produces a paraxial image at the vertex of the mirror.

Here the paraxial object is located at the objects-space principal point $\mathbf{H}$, *which coincides with* $\mathbf{V}, \mathbf{V}'$, *and* $\mathbf{H}'$. *Again, the relationship between the angles of incidence and reflection for the paraxial ray shows that the rays appear to emerge from the same location.*



You have a bit of a problem (but only a bit!) when evaluating this image distance from the imaging equation because $s = 0$, *but you can solve it if you just plow through it:*

$$s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{-12.5\,\text{mm}} - \frac{1}{0\,\text{mm}}\right)^{-1} = (-\infty)^{-1} = \boxed{0 = s'}$$

(e) Determine the transverse magnification for the object-image combination in part (d).

The transverse magnification "appears to be"

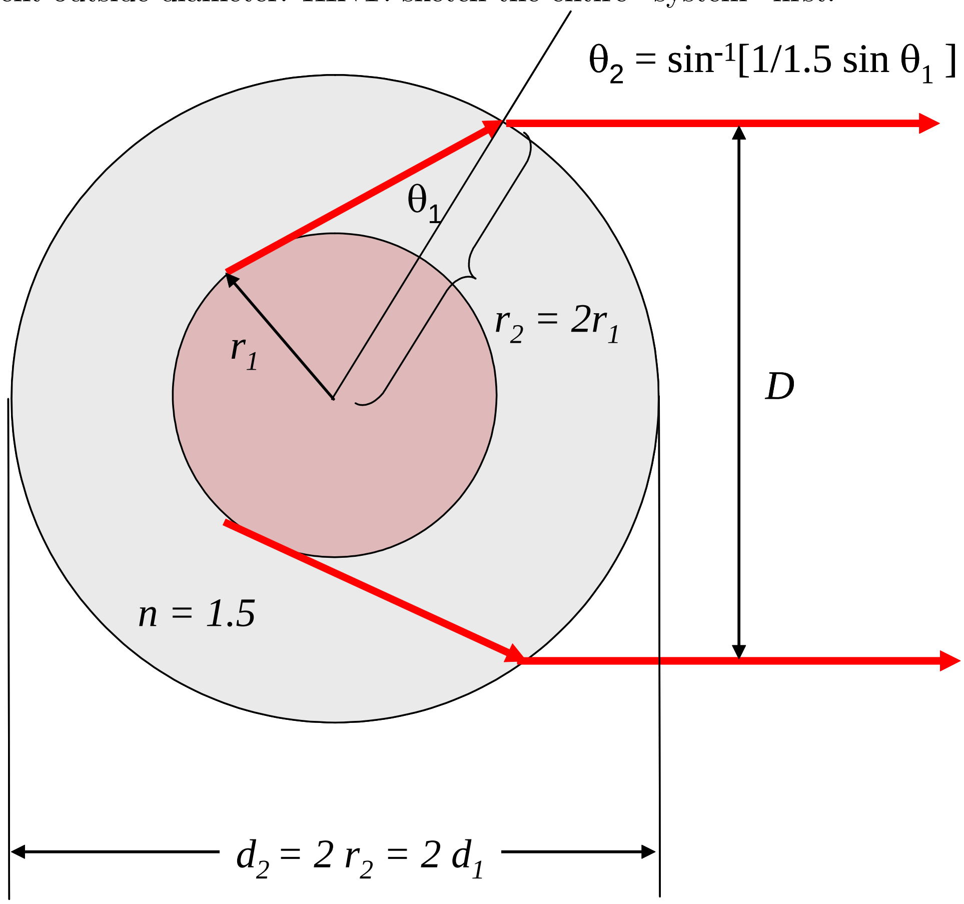
$$M_T = -\frac{s'}{s} = -\frac{0}{0} = ?$$

but what this really means is that the object and image are located at the object-space and image-space principal points,which coincide with each other and with the vertices,Therefore $\boxed{M_T = +1}$

(f) Sketch the object, system, and image in the configuration in part (d).

done in part (d)

3. A mercury thermometer is constructed from a cylindrical glass tube ($n = 1.5$). The outer diameter is twice as large as the inner diameter of the tube. The outer diameter is small (a few mm) and *much* smaller than the viewing distance; this means that the rays reaching the eye are approximately parallel. Determine the apparent diameter of the mercury column (i.e., the diameter of the inner wall of the glass tube) relative to the apparent outside diameter. HINT: sketch the entire "system" first.



$$n_1 = 1.5 \textit{ in Snell's law: } n_1 \sin[\theta_1] = n_2 \sin[\theta_2]$$

$$1.5 \cdot \sin[\theta_1] = 1 \cdot \sin[\theta_2] \implies \theta_2 = \sin^{-1}[1.5 \cdot \sin[\theta_1]]$$

$$\textit{From sketch: } \sin[\theta_1] = \frac{r_1}{r_2} = \frac{r_1}{2 \cdot r_1} = \frac{1}{2}$$

$$\implies \sin[\theta'] = 1.5 \cdot \frac{1}{2} \implies \theta' = \sin^{-1}\left[\frac{3}{4}\right] \cong 0.848 \textit{ radians } \cong 48.59°$$

$$D = 2 \cdot r_2 \sin[\theta'] = d_2 \cdot \sin\left[\sin^{-1}\left[\frac{3}{4}\right]\right] = \frac{3}{4} d_2$$

The observed diameter D of the column is equal to 3/4 the diameter of the outer tube.

4. Following is the prescription for a real multielement lens system with seven surfaces.

Surface↓	**Radius**	n'	t'
1	26.16 mm	1.6739	4.92 mm
2	1201.92 mm	1.0000	3.99 mm
3	−83.46 mm	1.6481	1.04 mm
4	25.67 mm	1.0000	5.53 mm
5	stop	1.0000	5.40 mm
6	302.57 mm	1.6515	2.57 mm
7	−54.79 mm	1.0000	-

(a) Calculate the vertex-to-vertex matrix for the system.

$$\varphi_1 = \frac{n'-n}{R_1} = \frac{1.6739-1}{26.16\,\text{mm}} \cong +0.025761\,\text{mm}^{-1}$$

$$\varphi_2 = \frac{n'-n}{R_1} = \frac{1-1.6739}{1201.92\,\text{m}} \cong -0.00056069\,\text{mm}^{-1}$$

$$\varphi_3 = \frac{n'-n}{R_1} = \frac{1.6481-1}{-83.46\,\text{mm}} \cong -0.0077654\,\text{mm}^{-1}$$

$$\varphi_4 = \frac{n'-n}{R_4} = \frac{1-1.6481}{+25.67\,\text{mm}} \cong -0.025247\,\text{mm}^{-1}$$

$$STOP \quad : \quad \varphi_5 = \frac{n'-n}{R_5} = \frac{1-1}{+\infty\,\text{mm}} \cong 0\,\text{mm}^{-1}$$

$$\varphi_6 = \frac{n'-n}{R_6} = \frac{1.6515-1}{+302.57\,\text{mm}} \cong +0.002153\,\text{mm}^{-1}$$

$$\varphi_7 = \frac{n'-n}{R_7} = \frac{1-1.6515}{-54.79\,\text{mm}} \cong +0.011891\,\text{mm}^{-1}$$

$$\mathcal{R}_1 = \begin{bmatrix} 1 & 0 \\ -\varphi_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.025761\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_1 = \begin{bmatrix} 1 & \frac{t'}{n'} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4.92\,\text{mm}}{1.6739} \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 2.9392\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_2 = \begin{bmatrix} 1 & 0 \\ +0.00056069\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_2 = \begin{bmatrix} 1 & \frac{3.99\,\text{mm}}{1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3.99\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_3 = \begin{bmatrix} 1 & 0 \\ +0.0077654\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_3 = \begin{bmatrix} 1 & \frac{1.04\,\text{mm}}{1.6481} \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0.63103\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_4 = \begin{bmatrix} 1 & 0 \\ +0.025247\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_4 = \begin{bmatrix} 1 & \frac{5.53\,\text{mm}}{1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5.53\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_5 = \begin{bmatrix} 1 & 0 \\ 0\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_5 = \begin{bmatrix} 1 & \frac{5.40\,\text{mm}}{1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5.40\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_6 = \begin{bmatrix} 1 & 0 \\ -0.002153\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{T}_6 = \begin{bmatrix} 1 & \frac{2.57\,\text{mm}}{1.6515} \\ 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 1.5562\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$\mathcal{R}_7 = \begin{bmatrix} 1 & 0 \\ -0.011891\,\text{mm}^{-1} & 1 \end{bmatrix}$$

$$\mathcal{M}_{\mathbf{VV'}} = \mathcal{R}_7\mathcal{T}_6\mathcal{R}_6\mathcal{T}_5\mathcal{R}_5\mathcal{T}_4\mathcal{R}_4\mathcal{T}_3\mathcal{R}_3\mathcal{T}_2\mathcal{R}_2\mathcal{T}_1\mathcal{R}_1$$

$$\boxed{\mathcal{M}_{\mathbf{VV'}} = \begin{bmatrix} 0.829\,43 & 23.106\,\text{mm} \\ -0.010004\,\text{mm}^{-1} & 0.92695 \end{bmatrix}}$$

$$\textit{Check determinant} \quad : \quad \det \begin{bmatrix} 0.82943 & 23.106\,\text{mm} \\ -0.010004\,\text{mm}^{-1} & 0.92695 \end{bmatrix} = 0.999\,99 \cong 1$$

(b) Locate and determine the magnification of an image of an object located 1000 mm in "front" of the front vertex (so that $\overline{\mathbf{OV}} = 1000\,\text{mm}$).

$$\textit{ray vector at object has height 0} \quad : \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathcal{T}_{\mathbf{OV}} = \begin{bmatrix} 1 & 1000\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

output ray vector :

$$\mathcal{M}_{\mathbf{VV'}}\mathcal{T}_{\mathbf{OV}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} y' \\ n'u' \end{bmatrix}$$

$$= \begin{bmatrix} 0.829\,43 & 23.106\,\text{mm} \\ -0.010004\,\text{mm}^{-1} & 0.92695 \end{bmatrix} \begin{bmatrix} 1 & 1000\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 852.54\,\text{mm} \\ -9.077\,1 \end{bmatrix}$$

$$\textit{distance to image } \overline{\mathbf{V'O'}} = -\frac{y'}{n'u'} = -\frac{852.54\,\text{mm}}{-9.077\,1} = \boxed{\overline{\mathbf{V'O'}} = +93.922\,\text{mm}}$$

Could find magnification by evaluating the "conjugate-to-conjugate matrix":

$$\mathcal{T}_{\mathbf{V'O'}}\mathcal{M}_{\mathbf{VV'}}\mathcal{T}_{\mathbf{OV}} = \begin{bmatrix} M_T & 0\,\text{mm} \\ 0\,\text{mm}^{-1} & M_T^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & +93.922\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.829\,43 & 23.106\,\text{mm} \\ -0.010004\,\text{mm}^{-1} & 0.92695 \end{bmatrix} \begin{bmatrix} 1 & 1000\,\text{mm} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.110\,17 & 1.309\,9\times 10^{-3}\,\text{mm} \\ -\frac{1.000\,4\times10^{-2}}{\text{mm}} & -9.077\,1 \end{bmatrix} \to \begin{bmatrix} -0.11017 & 0\,\text{mm} \\ 0\,\text{mm}^{-1} & -9.077\,1 \end{bmatrix}$$

$$\Longrightarrow \boxed{M_T \cong -0.11}$$

(c) Locate the cardinal points

$$\mathcal{M}_{\mathbf{VV'}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Longrightarrow f_{eff} = \overline{\mathbf{H'F'}} = \overline{\mathbf{FH}} = -\frac{1}{C} = -\frac{1}{-0.010004\,\text{mm}^{-1}}$$

$$\boxed{f_{eff} = \overline{\mathbf{H'F'}} = \overline{\mathbf{FH}} = 99.96\,\text{mm}}$$

$$BFD = \overline{\mathbf{V'F'}} = -\frac{A}{C} = -\frac{0.829\,43}{-0.010004\,\text{mm}^{-1}} = \boxed{\overline{\mathbf{V'F'}} \cong 82.91\,\text{mm}}$$

$$FFD = \overline{\mathbf{FV}} = -\frac{D}{C} = -\frac{0.92695}{-0.010004\,\text{mm}^{-1}} = \boxed{\overline{\mathbf{FV}} \cong 92.66\,\text{mm}}$$

$$\overline{\mathbf{H'V'}} = \frac{A-1}{C} = \frac{0.829\,43 - 1}{-0.010004\,\text{mm}^{-1}} = \boxed{\overline{\mathbf{H'V'}} \cong 17.05\,\text{mm}}$$

$$\overline{\mathbf{VH}} = \frac{D-1}{C} = \frac{0.92695 - 1}{-0.010004\,\text{mm}^{-1}} = \boxed{\overline{\mathbf{VH}} \cong 7.30\,\text{mm}}$$

(d) Locate the entrance and exit pupils (HINT: pupils are images of the stop!)

The idea here is to locate the image of the stop, so you have to evaluate an "object-to-image" matrix that is composed of subsets of the vertex-to-vertex matrix. Trace the object ray "backward" from the stop to locate the place where a ray through the center of the stop with arbitrary ray angle (call it 1) crosses the axis

$$\begin{bmatrix} 1 & 0 \\ -0.011891\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{2.57\,\text{mm}}{1.6515} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.002153\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5.40\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6.938\,1\,\text{mm} \\ 0.905\,87 \end{bmatrix}$$

so a ray from the center of the stop with angle 1 exits the image-space vertex at a height of 6.9381 mm *with a ray angle of* +0.905 7 *radians. The distance from the vertex to the exit pupil is:*

$$\overline{\mathbf{V'X}} = -\frac{y'}{n'u'} = -\frac{6.9381\,\text{mm}}{0.905\,87} = \boxed{\overline{\mathbf{V'X}} \cong -7.659\,\text{mm}}$$

which indicates that the image of the stop in image space (the exit pupil) is virtual

Do the same thing for the entrance pupil; I'll "turn the system around." The output ray vector is:

$$\begin{bmatrix} 1 & 0 \\ -0.025761\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2.9392\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ +0.00056069\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3.99\,\text{mm} \\ 0 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 \\ +0.0077654\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.63103\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ +0.025247\,\text{mm}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 5.53\,\text{mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 14.5\,\text{mm} \\ 0.820\,77 \end{bmatrix}$$

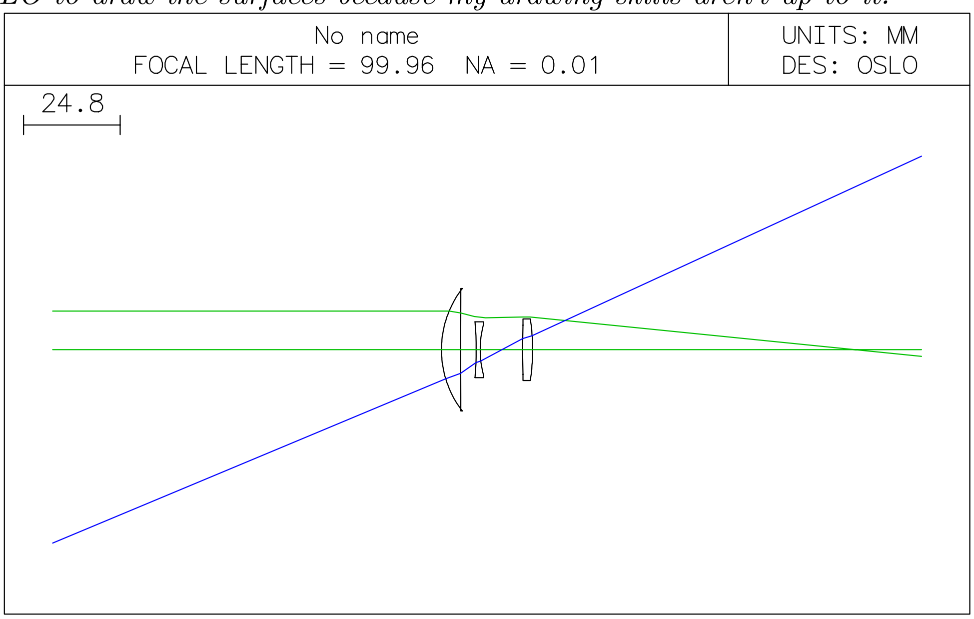
so a ray from the center of the stop with angle 1 exits the OBJECT-space vertex at a height of 14.5 mm *with a ray angle of* +0.820 77 *radians. The distance from the OBJECT-SPACE vertex to the ENTRANCE pupil is:*

$$\overline{\mathbf{VN}} = -\frac{y'}{n'u'} = -\frac{14.5\,\text{mm}}{0.820\,77} \cong -17.666\,\text{mm} \implies \overline{\mathbf{NV}} = -\overline{\mathbf{VN}} = \boxed{\overline{\mathbf{NV}} \cong +17.666\,\text{mm}}$$

The Entrance Pupil is real.

(e) Draw the system to scale showing provisional marginal and chief rays (the chief ray should go through the center of the stop!)

I used OSLO to draw the surfaces because my drawing skills aren't up to it!



Provisional ray trace, marginal ray in green, chief ray in blue.

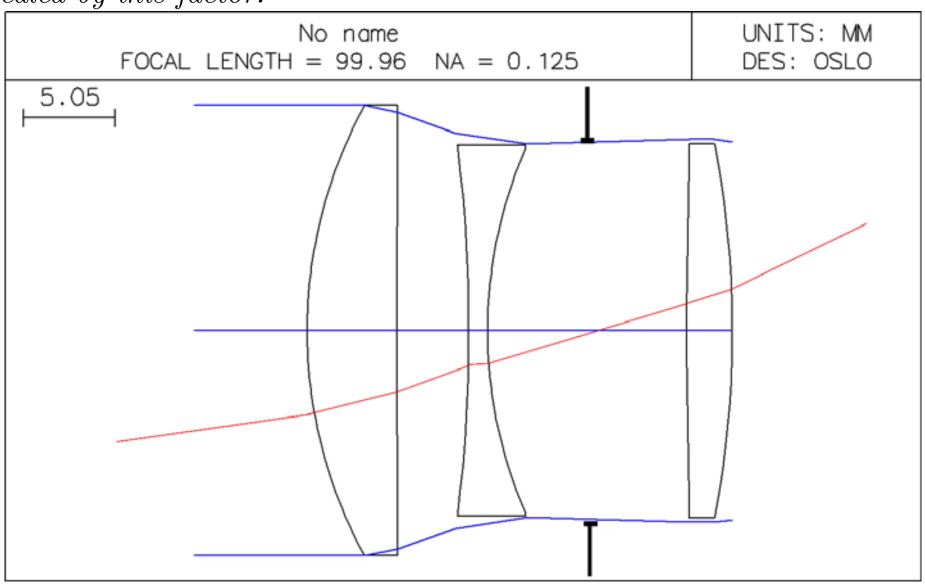
(f) (OPTIONAL, Extra Credit). If the lens system is known to have an f/number of 4 and the full field of view is 28°, determine the actual marginal and chief rays for this system if the object is at ∞.

You need to scale the chief ray angle (the ray through the center of the stop) so that the angle at the object side is 14°. Since the assumed chief ray angle at the stop was 1, then all chief ray heights and angles are scaled by $14 \div \left(\frac{180}{\pi}\right) \cong 0.244$ *because all equations are linear.*

The f/number allows you to scale the marginal ray height so that the diameter of the entrance pupil is the correct value. The f/# is:

$$\begin{aligned} \mathrm{f}/\# &= \frac{f}{D} \Longrightarrow D = \frac{f}{\mathrm{f}/\#} = \frac{99.96\,\mathrm{mm}}{4} = 24.99\,\mathrm{mm} \\ &\Longrightarrow y = \frac{D}{2} = \frac{24.99\,\mathrm{mm}}{2} = 12.495\,\mathrm{mm} \end{aligned}$$

Since the "provisional" marginal ray height was 1, all marginal ray heights and angles are scaled by this factor.



Final ray trace from OSLO with marginal ray in blue and chief ray in red.

5. The index of refraction of a hypothetical material is found to vary in proportion to the reciprocal of the vacuum wavelength λ_0. Determine the modulation ("group") velocity in terms of the average ("phase") velocity at a given wavelength. Does the material exhibit normal or anomalous dispersion?

$$n[\lambda_0] = \frac{A}{\lambda_0} \implies \lambda_0 = \frac{A}{n[\lambda]}$$

Right away we can see that n decreases as λ increases, so we expect that the dispersion is normal

$$\begin{aligned}
\textit{wavelength in matter is } \lambda' &\equiv \frac{\lambda_0}{n[\lambda]} = \frac{\lambda_0}{\left(\frac{A}{\lambda_0}\right)} = \frac{\lambda_0^2}{A} \\
k &= \frac{2\pi}{\lambda'} = \frac{2\pi n}{\lambda_0} = \frac{2\pi n^2}{A} \implies \boxed{n = \sqrt{\frac{Ak}{2\pi}}} \\
\mathrm{v}_\phi &= \frac{\omega}{k} = \frac{c}{n} \implies \omega = \frac{ck}{n} \\
\mathrm{v}_{\mathrm{mod}} &= \frac{d\omega}{dk} = c\frac{d}{dk}\left(\frac{k}{n}\right) = c \cdot \left(\frac{1}{n} - \frac{k}{n^2}\frac{dn}{dk}\right) \\
&= c \cdot \left(\frac{1}{n} - \frac{k}{n^2}\frac{dn}{dk}\right) = \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right) \\
&\boxed{\mathrm{v}_{\mathrm{mod}} = \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)}
\end{aligned}$$

$$\begin{aligned}
n &= \sqrt{\frac{Ak}{2\pi}} \implies \boxed{\frac{dn}{dk} = \frac{1}{2}\sqrt{\frac{A}{2\pi k}}} \\
\frac{k}{n} &= \frac{k}{\sqrt{\frac{Ak}{2\pi}}} = \sqrt{\frac{2\pi}{Ak}}k = \sqrt{\frac{2\pi k}{A}}
\end{aligned}$$

: *Substitute into expression for* $\mathrm{v}_{\mathrm{mod}}$

$$\begin{aligned}
\mathrm{v}_{\mathrm{mod}} &= \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right) = \mathrm{v}_\phi\left(1 - \sqrt{\frac{2\pi k}{A}} \cdot \frac{1}{2} \cdot \sqrt{\frac{A}{2\pi k}}\right) \\
&= \mathrm{v}_\phi\left(1 - \frac{1}{2}\right) \\
&\boxed{\mathrm{v}_{\mathrm{mod}} = \frac{1}{2}\mathrm{v}_\phi}
\end{aligned}$$

Thus the modulation velocity (or the "group velocity") is one half of the phase velocity! The dispersion is $\boxed{\textit{normal}}$, *as expected.*

6. In a diffraction experiment, a pinhole ("point") source with $\lambda = 600\,\mathrm{nm}$ is used. The distance from this source to a diffracting aperture is $10\,\mathrm{m}$ and the aperture is a circular hole of diameter $1\,\mathrm{mm}$. The light diffracted from the circular aperture is observed on a screen located at a variable distance from the aperture. We know that the observed diffraction falls into the category of "Fraunhofer" if the distance from the aperture to the screen is "large" and "Fresnel" if the distance is "small".

   (a) Make a case for a distance from the aperture to the screen that roughly divides between the two classes of diffraction. Note that there is no one "correct" answer; it's your argument that matters.

   The diffracting aperture may be represented by the function:

$$f[x,y] \to f(r) = CYL\left(\frac{r}{d}\right) = CYL\left(\frac{r}{1\,\mathrm{mm}}\right)$$

   For completeness, consider the assumption that the illuminating wavefront at the diffracting aperture is planar. The distance from the point source to the diffracting aperture is $z_0 = 10\,\mathrm{m}$, *so the expanding spherical wave propagates for many wavelengths* N:

$$N = \frac{z_0}{\lambda_0} = \frac{10\,\mathrm{m}}{600\,\mathrm{nm}} \cong 1.67 \times 10^7 \textit{ wavelengths}$$

   Between the center and edge of the diffracting aperture, the phase of the wavefront will change by: $\frac{\left(\frac{1}{2}\,\mathrm{mm}\right)^2}{20\,\mathrm{m}} = 1.25 \times 10^{-5}\,\mathrm{mm} = \frac{1}{40\,\mathrm{m}}\,\mathrm{mm} = 2.5 \times 10^{-5}$ *radians, which is pretty flat.*

$$\begin{aligned}\Delta\phi &= \frac{\left(\frac{1}{2}\,\mathrm{mm}\right)^2}{20\,\mathrm{m}\cdot 600\,\mathrm{nm}} \cong 2.08 \times 10^{-2} \textit{ radians} \cong 1.2^\circ \\ \cos\left[2.08 \times 10^{-2} \textit{ radians}\right] &\cong 0.99978 \cong 1\end{aligned}$$

   :*so the assumption that light is a plane wave is a good one*

   The approximation for the distance $|\underline{\mathbf{r}}|$ *in the diffraction integral is:*

$$\begin{aligned}|\underline{\mathbf{r}}| &= z_1 \cdot \left(1 + \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2}\right)^{\frac{1}{2}} \\ &= z_1\left(1 + \frac{1}{2}\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{z_1^2} - \frac{1}{8}\frac{\left((x_1 - x_0)^2 + (y_1 - y_0)^2\right)^2}{z_1^4} + \cdots\right)\end{aligned}$$

   In the Fresnel region, the the third (and higher-order) terms are approximately zero. In the Fraunhofer region, the second-order term is assumed to be approximately zero:

$$\cos\left[\frac{(x_0^2 + y_0^2)}{\lambda_0 z_1}\right] = \cos\left[\frac{r_0^2}{\lambda_0 z_1}\right] \cong 1$$

   The question is really asking how small this factor needs to be. If we wanted the cosine of this angle to be at least 0.95 then this ratio needs to be no larger than 0.3:

$$\frac{r_0^2}{\lambda_0 z_1} \lessapprox 0.3 \implies z_1 \gtrapprox \frac{(0.5\,\mathrm{mm})^2}{600\,\mathrm{nm}\cdot 0.3} \cong 1.39\,\mathrm{m}$$

(b) Sketch an approximation of the pattern observed if the distance from the aperture to the screen is 2 m. Information from the courses on Linear Mathematics may be helpful..

By the criterion just evaluated, this distance is in the Fraunhofer region, so the diffraction pattern is a scaled replica of the squared magnitude of the Fourier transform of the diffracting aperture function. This Fourier transform is:

$$\mathcal{F}_2\left\{CYL\left(\frac{r}{1\,\text{mm}}\right)\right\} = \frac{\pi\cdot(1\,\text{mm})^2}{4} SOMB\,(1\,\text{mm}\cdot\rho)$$

The Fraunhofer diffraction pattern is a scaled replica, where the scaling factor is $\lambda_0 z_1$:

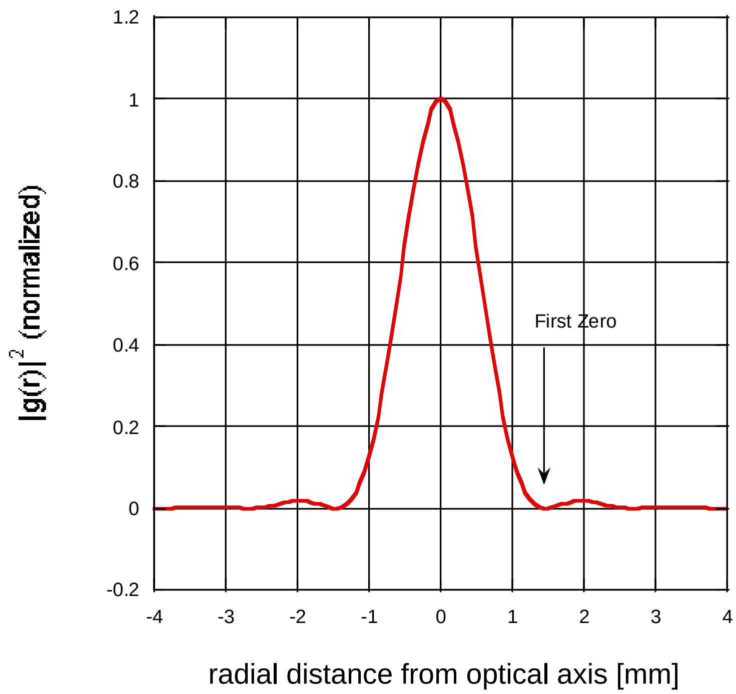
$$g(r_1) \propto \mathcal{F}_2\{f(r)\}|_{\rho=\frac{r}{\lambda_0 z_1}} \propto SOMB\left(\frac{d\cdot\rho}{\lambda_0\cdot z_1}\right)$$

$$= SOMB\left(\frac{\rho}{\left(\frac{\lambda_0\cdot z_1}{d}\right)}\right) = SOMB\left(\frac{\rho}{\frac{600\,\text{nm}\cdot 2\,\text{m}}{1\,\text{mm}}}\right) = SOMB\left(\frac{\rho}{1.2\,\text{mm}}\right)$$

This is a "sombrero" or "besinc" function with width parameter 1.2 mm, *and so the irradiance is the squared magnitude:*

$$|g(r_1)|^2 \propto SOMB^2\left(\frac{\rho}{1.2\,\text{mm}}\right)$$

The first zero is located at approximately $1.2\,\text{mm}\cdot 1.22 = 1.46\,\text{mm}$ *from the center:*



(c) Consider that there are two identical point sources separated by a distance d at the source plane and the pattern is observed at the same distance of 2 m in part (b). Determine the value of d such that the two sources just be "resolved" at the observation plane.

Again, you have some freedom to choose the metric for resolution. In this case where there is only a single wavelength so that the light is "coherent", the commonly used distance from the peak to the first zero is not large enough to ensure that the images are distinguishable.