SIMG-712-90 FINAL EXAM SOLUTIONS

1. You are to design an imaging system for use in spacebased astronomy in low earth orbit – this is just to ensure that there are no atmospheric aberrations. The sensor is a pixelated CCD whose square pixels are $9\,\mu\text{m} \times 9\,\mu\text{m}$ that butt against each other so that all photons onto the sensor are imaged (there is no "dead" area between CCD pixels). The diameter of the optic is d mm and the focal length is f mm..

   (a) Determine the f/# of the system such that the diameter of the central core of the diffraction spot (the circle enclosed by the first zero of the Fraunhofer diffraction pattern) just fills a CCD pixel for $\lambda = 550\,\text{nm}$.

   Use the diffraction formula in the Fraunhofer region. The stop of the system is assumed to be circular (since the lens is assumed to be circular).

$$D \cong 2.44\frac{\lambda f}{d} = 2.44 \cdot \lambda \cdot \text{f}/\#$$

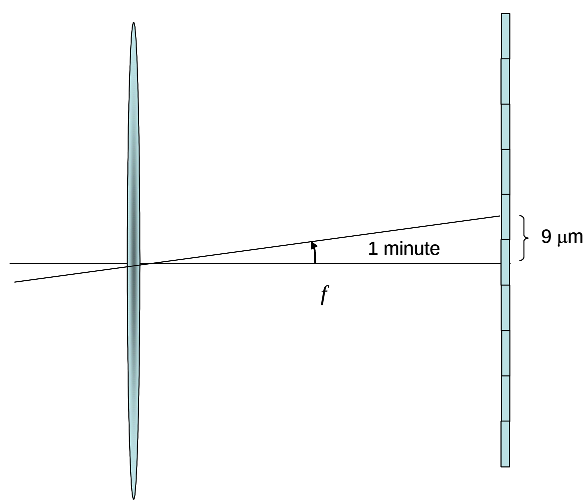
$$\Longrightarrow \text{f}/\# \cong \frac{D}{2.44\lambda} = \frac{9\,\mu\text{m}}{2.44 \cdot 550\,\text{nm}} = \frac{9000}{1342} \cong \boxed{6.7 \cong \text{f}/\#}$$

   (b) Determine the focal length of the system that results in an "angular resolution" (also called a "plate scale") of 1 arcsecond per pixel at the same wavelength. Use this result to determine the diameter of the optic that also satisfies the condition in part (a).

$$1 \text{ arcsecond} = \frac{1}{60}\text{ arcminute} = \frac{1}{3600}^{\circ} = \frac{1}{3600}^{\circ} \cdot \frac{\pi}{180}\frac{\text{radians}}{^{\circ}} \cong 4.85 \cdot 10^{-6}\text{ radians}$$

$$= 4.85\ \mu\text{radians}$$

   We want the pixel "pitch" (distance between pixel centers) to correspond to this angle of $4.85\ \mu$radians.



   The ray through the center of the lens is not deflected, so we can immediately eval-

uate the focal length by trigonometry. Since the angle is small, we have:

$$
\begin{aligned}
f\,[\mu\mathrm{m}] \cdot 4.85 \cdot 10^{-6}\ \text{radians} &= 9\,\mu\mathrm{m} \\
&\Longrightarrow f = \frac{9\,\mu\mathrm{m}}{4.85 \cdot 10^{-6}} \cong 1,856\,\mathrm{mm} = \boxed{f \cong 1.856\,\mathrm{m} \cong 73\,\mathrm{in}} \\
\mathrm{f}/6.7 &\Longrightarrow d = \frac{f}{\mathrm{f}/\#} \\
&\Longrightarrow d \cong \frac{1856\,\mathrm{mm}}{6.7} \cong \boxed{277\,\mathrm{mm} \cong 10.9\,\mathrm{in} \cong d}
\end{aligned}
$$

(c) For the same diameter and focal length, determine the size of the diffraction spot and the plate scale for light with $\lambda = 400\,\mathrm{nm}$ and $\lambda = 1000\,\mathrm{nm}$.

$$
\begin{aligned}
\lambda &= 400\,\mathrm{nm} \Longrightarrow D \cong 2.44 \cdot 400\,\mathrm{nm} \cdot 6.7 \cong \boxed{6.5\,\mu\mathrm{m} < \text{pixel size}} \\
\lambda &= 1000\,\mathrm{nm} \Longrightarrow D \cong 2.44 \cdot 1000\,\mathrm{nm} \cdot 6.7 \cong \boxed{16.3\,\mu\mathrm{m} > \text{pixel size}}
\end{aligned}
$$

2. The area density of cone sensors in the fovea of the eye is approximately $160,000\,\text{mm}^{-2}$. For imaging in bright light, assume that the diameter of the eye's pupil is approximately 2 mm.

   (a) Determine the approximate angular resolution of the eye when viewing an object at ∞. Assume that the wavelength $\lambda = 500\,\text{nm}$. This problem uses the eye model that was used to illustrate ray tracing on p. 203 of the most recent notes (dated 2/12/2005).

   One criterion for resolution is Rayleigh's. If no aberrations, the half-angle can be used as a measure of the distance between "pixels":

$$\begin{aligned}\theta &\cong 1.22\frac{\lambda}{d} = 1.22 \cdot \frac{500\,\text{nm}}{2\,\text{mm}} \cong 3.05 \times 10^{-4}\text{ radians} \cong 0.305\text{ milliradians} \\ &\cong 3.05 \times 10^{-4}\text{ radians} \cdot 206265\frac{\textit{seconds}}{\textit{radian}} \cong 63\ \textit{arcseconds}\end{aligned}$$

   (b) Determine if the image is sampled adequately or undersampled by the cones in the fovea (assume that the lens of the eye has no significant aberrations).

   Given that the focal length of the eye is approximately 17 mm (*from the eye model in the notes), we can find the diameter of the diffraction spot on the retina:*

$$f \cong 17\,\text{mm} \Longrightarrow D \cong 2 \cdot 17\,\text{mm} \cdot 3.05 \times 10^{-4} \cong 10.4\,\mu\text{m}$$

   If area density is $160,000\,\text{mm}^{-2}$*, the linear density is approximately* $\sqrt{160,000} = 400\,\text{mm}^{-1}$. *Thus the linear dimension of a receptor is:*

$$\ell \cong \frac{1}{400}\text{mm} \Longrightarrow \boxed{2.5\,\mu\text{m}}$$

   Thus there are approximately 4 receptors within the diameter of the diffraction spot, which leads us to expect that the image is "oversampled". NOTE: this did not account for neural connections (the neural "net") that combine signals from cones and thus "effectively" increase the size of receptors.

   (c) Describe in words the changes in the optics of the eye as the object distance is decreased from ∞ to the near point (the closest distance where the eye lens can focus)

   The power of the lens of the eye increases to focus more closely. This is accomplished by relaxing the ciliary muscles to allow the lens to "thicken". Of course, as we grow older, the lens loses flexibility and cannot as easily accommodate.

   (d) If the distance from the near point to the eye is the "standard distance" of 10" $\cong$ 250 mm, if the distance from the cornea to the retina remains fixed at the distance for distant vision (object at ∞), and if the two surfaces of the eye lens have equal and opposite curvatures regardless of the object distance, determine the radii of curvature of the surfaces of the eye lens that are necessary to position the image on the retina.

   Eye model

R		$+7.8\,\mathrm{mm}$	$+R$	$-R$
t'		$3.6\,\mathrm{mm}$	$3.6\,\mathrm{mm}$	
n'	1.0	1.336	1.413	1.336
$-\varphi = -\frac{n'-n}{R}$		$-0.043077\,\mathrm{mm}^{-1}$	$-\frac{0.007700}{R}\,\mathrm{mm}^{-1}$	$-\frac{0.012833}{R}\,\mathrm{mm}$
$\frac{t'}{n'}$		$\frac{3.6\,\mathrm{mm}}{1.336} = 2.694611\,\mathrm{mm}$	$\frac{3.6\,\mathrm{mm}}{1.413} = 2.54771\,\mathrm{mm}$	$\boxed{12.699\ \mathrm{mm}}$

$$f = 12.699\,\mathrm{mm} \cdot n' = 12.699 \cdot 1.336 \cong 16.966\,\mathrm{mm}$$

If $s' \cong 17\,\mathrm{mm}$ *and* $s = 250\,\mathrm{mm}$*, we can find the required focal length to form the image:*

$$f = \left(\frac{1}{250\,\mathrm{mm}} + \frac{1}{17\,\mathrm{mm}}\right)^{-1} \cong \boxed{15.9\,\mathrm{mm} \cong f}$$

The curvatures of the lens surface in the model are $R_1 = +10\,\mathrm{mm}, R_2 = -6\,\mathrm{mm}$*, which are not equal. The index of the lens is modeled as* $n = 1.413$ *and the index of the surrounding medium (the "humour") is* $n = 1.336$ *(approximately the same as water). If we assume the curvatures are equal (which they seem unlikely to be), then the approximate radii can be evaluated via the lensmaker's equation (that assumes that the lens is thin):*

$$\begin{aligned}
\frac{1}{f} &= \left(\frac{n_l}{n_s} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\
\frac{1}{15.9\,\mathrm{mm}} &= \left(\frac{1.413}{1.336} - 1\right)\left(\frac{1}{R_1} - \frac{1}{-R_1}\right) = 0.0576\left(\frac{2}{R_1}\right) \\
&\implies R_1 = 0.0576 \cdot 2 \cdot 15.9\,\mathrm{mm} \cong \boxed{1.83\,\mathrm{mm} \cong R}
\end{aligned}$$

which is much shorter than the lens when focused at ∞*. Note that if we want to account for the thickness of the lens, we have a third variable in the equation that changes with the radii and the problem is underdetermined. To get somewhat closer to the actual answer, we could assume the same thickness as for the unaccommodated eye and use the thick-lens equation:*

$$\begin{aligned}
\frac{1}{f} &= (n_l - n_s)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - n_s)\,t}{n_l R_1 R_2}\right) \\
\to \frac{1}{f} &= (n_l - n_s)\left(\frac{1}{R_1} - \frac{1}{-R_1} + \frac{(n_l - n_s)\,t}{n_l R_1\,(-R_1)}\right) \\
\frac{1}{15.9\,\mathrm{mm}} &= (0.077)\left(\frac{2}{R_1} - \frac{0.077 \cdot 3.6\,\mathrm{mm}}{1.413 \cdot R_1^2}\right) \\
\frac{1}{15.9\,\mathrm{mm}} &= \frac{-1.5106 \times 10^{-2}\,\mathrm{mm}}{R_1^2} + \frac{0.154}{R_1} \\
\left(6.2893 \times 10^{-2}\,\mathrm{mm}^{-1}\right) R_1^2 &+ (-0.154)\,R_1 + 1.5106 \times 10^{-2}\,\mathrm{mm} = 0
\end{aligned}$$

which has two solutions:

$$R_1 = 0.010\,\mathrm{mm}, \boxed{2.34\,\mathrm{mm}}$$

The latter solution is more reasonable and is close to that from the thin lens equation. We also could try to iterate towards the solution.

(e) Assuming that a page of printed text is held at the "near point," determine the smallest linear dimension that the eye can read clearly given the density of cones and the pupil diameter.

If we assume the same pupil diameter of 2 mm *and the same wavelength of* 500 nm, *we get the same value of the "half-angle" of resolution using Rayleigh's criterion:*

$$\theta \cong 1.22\frac{\lambda}{d} \cong 0.305 \text{ milliradians}$$

at a distance of 200 mm, *this angle corresponds to a physical dimension for the diameter of the diffraction spot of:*

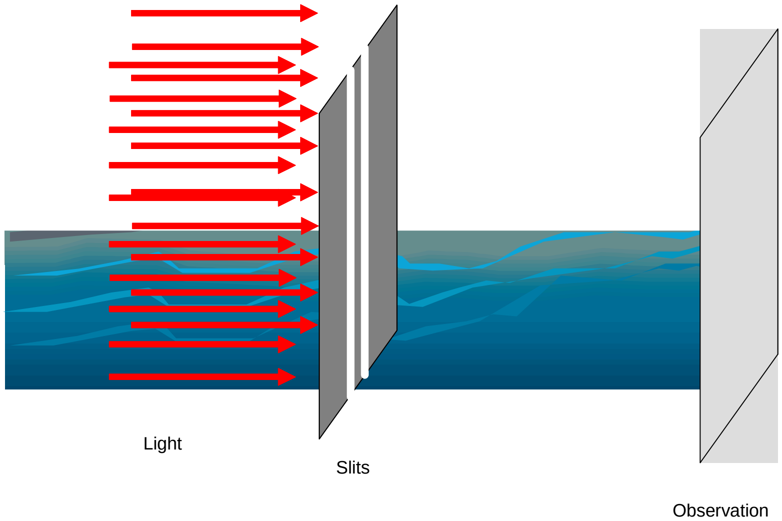
$$h \cong 2 \cdot 0.305 \cdot 10^{-3} \cdot 200\,\text{mm} \cong 122\,\mu\text{m}$$

But we need to see several samples across a text character to read it – I would say at least 5 and more like 10. If the latter, then the smallest character size we might expect to read is

$$10 \cdot 122\,\mu\text{m} = 1.22\,\text{mm}$$

which seems about right.

3. A double slit experiment is performed with the slits both above and under a water surface ($n = 1.33$). Suppose that the separation of the slits is 0.35 mm, while the temporal frequency of the light is $\nu = 5 \times 10^{14}$ Hz. The distance from the slits to the observation screen is 2 m.



(a) Describe the differences (if any) between the patterns observed on the screen under water and in air.

The wavelength changes in water. Since the physical distance between the slits does not change, there is a change in the optical path difference in water relative to air.. Since $\nu = 5 \times 10^{14}$ Hz, *the wavelength in air is:*

$$\lambda_{air} = \frac{c}{\nu} = \frac{3 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{sec}}}{5 \times 10^{14}\,\mathrm{Hz}} = 600\,\mathrm{nm}$$

The wavelength in water is:

$$\lambda_{water} = \frac{\lambda_{air}}{n} = \frac{600\,\mathrm{nm}}{1.33} \cong 451.1\,\mathrm{nm}$$

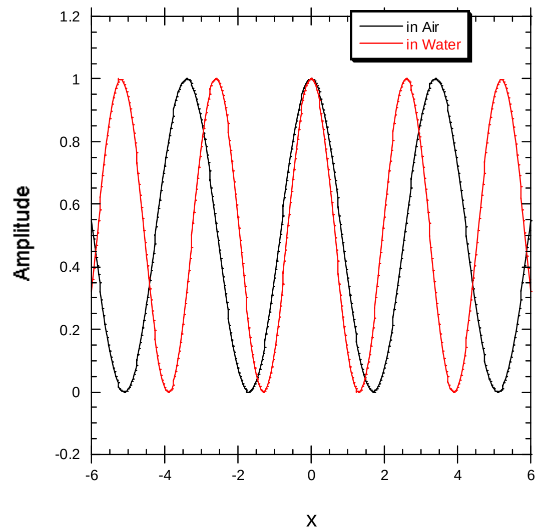
We know the expression for the period of the irradiance fringe:

$$D \cong \frac{L\lambda}{d} \cong \frac{2\,\mathrm{m}}{0.35\,\mathrm{mm}}\lambda \cong 5715\lambda$$

So the respective periods are:

$$\begin{aligned} D_{air} &\cong 5715 \cdot 600\,\mathrm{nm} \cong 3.4\,\mathrm{mm} \\ D_{water} &\cong 5715 \cdot 451.1\,\mathrm{nm} \cong 2.6\,\mathrm{mm} \end{aligned}$$

(b) Sketch the patterns in both media.



(c) Determine the separation on the observation screen between the central bright fringe and the second bright fringe.

Duh. This is just twice the period, so 6.8 mm *in air and* 5.2 mm *in water.*

4. A binary (or *bitonal*) image of size $N \times N$ pixels contains a rectangular region of level "1" surrounded by pixels with level "0."

(a) Determine the magnitude of the gradient of this image and the approximation obtained via the equation on bottom of p.411 (notes dated 2/12/2005):

$$\begin{aligned}|\boldsymbol{\nabla} f[n,m]| &= \sqrt{(\partial_x * f[n,m])^2 + (\partial_y * f[n,m])^2} \\ &\cong |\partial_x * f[n,m]| + |\partial_y * f[n,m]|\end{aligned}$$

We need to calculate the gradient for horizontal edges, vertical edges, and "corners". Calculate the two derivatives for each:

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$$\Longrightarrow \quad \frac{\partial f}{\partial x} = f[n+1,m] - f[n,m] =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$$\Longrightarrow \quad \frac{\partial f}{\partial y} = f[n,m+1] - f[n,m] =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	-1	-1	-1	-1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

The "first" definition of the magnitude of the gradient is:

$$\sqrt{(\partial_x * f[n,m])^2 + (\partial_y * f[n,m])^2} =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	+1	+1	+1	$+\sqrt{2}$	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

$$|\partial_x * f[n,m]| + |\partial_y * f[n,m]| =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	+1	+1	+1	+2	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

(b) Determine the histogram of edge "directions" computed from the azimuth angle of the gradient.

=

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	**0**	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{3\pi}{4}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	0	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	0	0
0	0	0	0	0	0	0	0

(c) Determine the values of the "isotropic" 3×3 Laplacian operator.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

*

1	1	1
1	-8	1
1	1	1

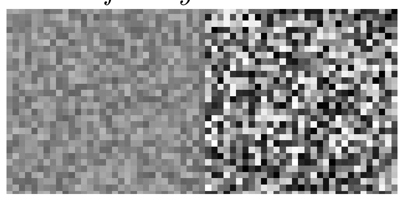
=

0	0	0	0	0	0	0	0
0	+1	+2	+3	+3	+2	+1	0
0	+2	-5	-3	-3	-5	+2	0
0	+3	-3	0	0	-3	+3	0
0	+3	-3	0	0	-3	+3	0
0	+2	-5	-3	-3	-5	+2	0
0	+1	+2	+3	+3	+2	+1	0
0	0	0	0	0	0	0	0

Sum of the values is zero, as expected.

5. A digital image $f[n,m]$ is created from random noise, i.e., random variations in gray level. There are two such areas in the image that have the same mean gray values but different variances. Describe how to segment the two regions by applying pixel (point) operators and linear shift-invariant local neighborhood operations (convolutions). In other words, the only nonlinear operation available to you is a pixel operation. Be sure to specify the sequence of operations and the kernels used in any convolutions.

   A possible histogram is the sum of the histograms of the component regions. The example shown was constructed from uniformly distributed noise (see below)

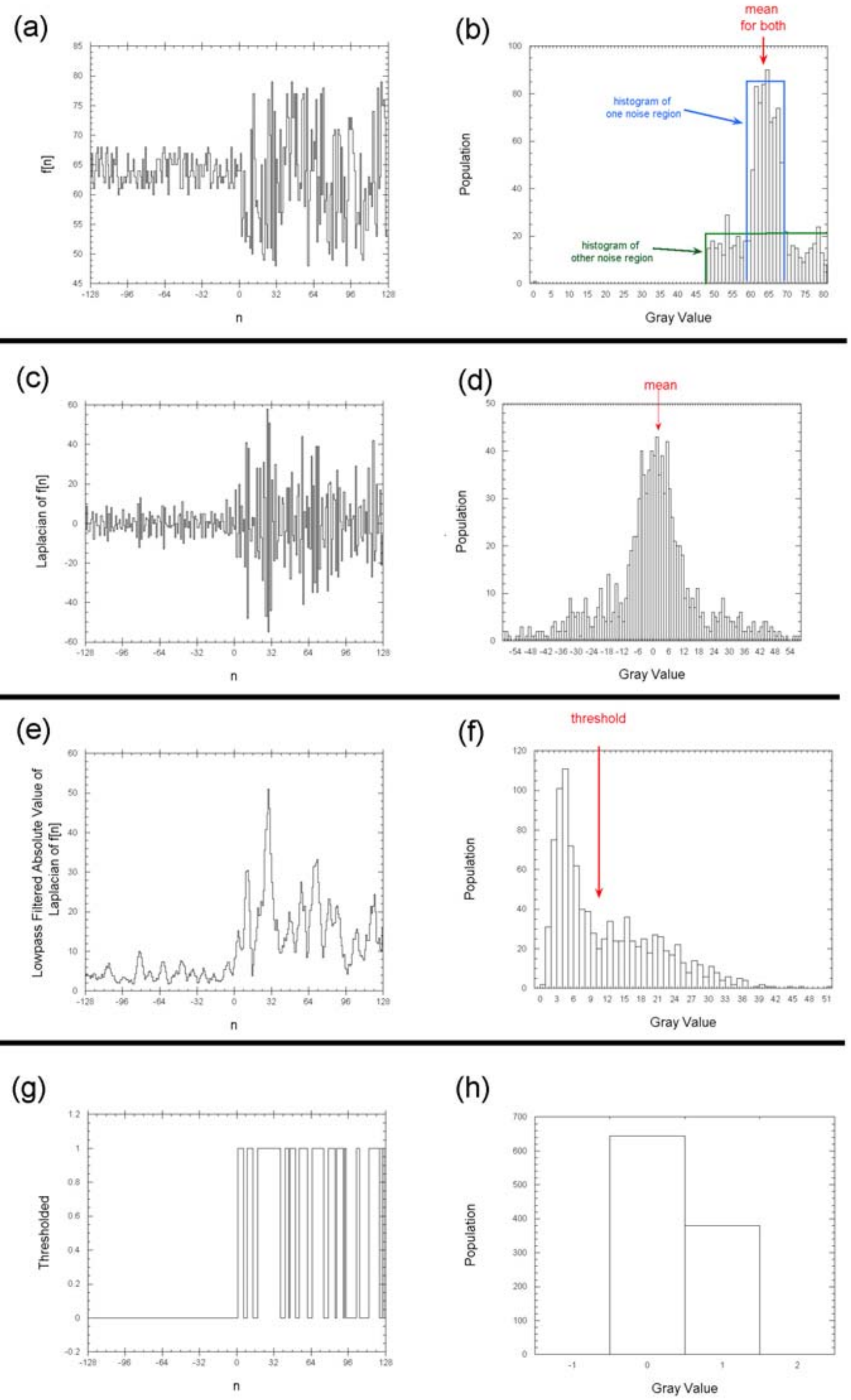


Magnified view of section of $f_q[n,m]$ near boundary between two regions of noise with the same mean and different variances.

   To convert "noisiness" to "gray level", use a differencing operator (highpass filter). I suggest using a flavor of Laplacian kernel, e.g.,

-1	-1	-1
-1	+8	-1
-1	-1	-1

   This blocks the constant part and amplifies the high-frequency noise. But the output is bipolar! So apply a nonlinear point operator of absolute value.



(a)
f[n]
n
(b)
mean
for both
histogram of
one noise region
histogram of
other noise region
Population
Gray Value
(c)
Laplacian of f[n]
n
(d)
mean
Population
Gray Value
(e)
Lowpass Filtered Absolute Value of
Laplacian of f[n]
n
(f)
threshold
Population
Gray Value
(g)
Thresholded
n
(h)
Population
Gray Value

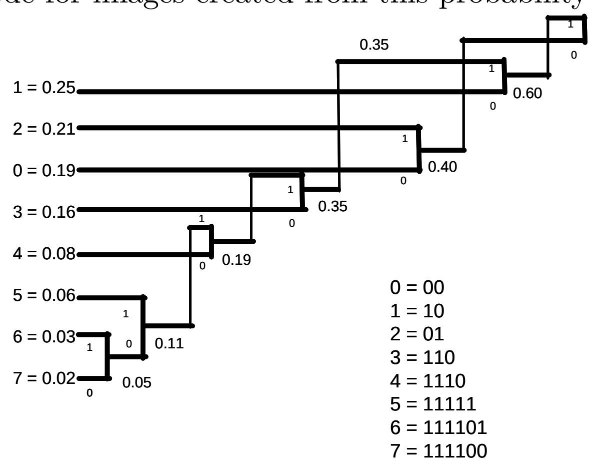
6. A digital image has 8 gray values selected from the probability distribution"

$$
\begin{aligned}
p[0] &= 0.19 \\
p[1] &= 0.25 \\
p[2] &= 0.21 \\
p[3] &= 0.16 \\
p[4] &= 0.08 \\
p[5] &= 0.06 \\
p[6] &= 0.03 \\
p[7] &= 0.02
\end{aligned}
$$

(a) Calculate the information content (also called the *entropy*) in BOTH binary digits and base-e digits (called *nats*) bits per pixel

$$
\begin{aligned}
I &= -\sum_{n=0}^{7} p_n \log_2 [p_n] \\
&\cong 2.651 \text{ bits per pixel} \\
&\cong \frac{2.651 \cdot \log[2]}{\log[e]} \cong 1.838 \text{ nats}
\end{aligned}
$$

(b) Construct a Huffman code for images created from this probability distribution.



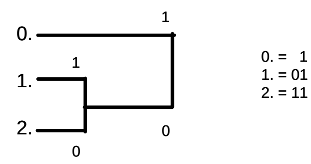
Note that the probability of 0.19 resulting from the sum of 0.11 (5,6,7) and 0.08 (4) could have been put above level 0 instead of below. The resulting code is slightly different.

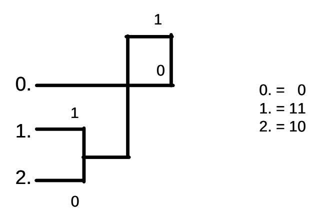
(c) Calculate the expected bit rate (bits per pixel) for an image from this distribution compressed using this Huffman code.

$2{\cdot}0.19+2{\cdot}0.25+2{\cdot}0.21+3{\cdot}0.16+4{\cdot}0.08+5{\cdot}0.06+6{\cdot}0.03+6{\cdot}0.02 = 2.7$ bits per pixel

(d) (OPTIONAL BONUS) How many unique Huffman codes exist for an image that consists of three different gray levels. Construct them.

The possible Huffman codes are:





The two codes are complements of each other