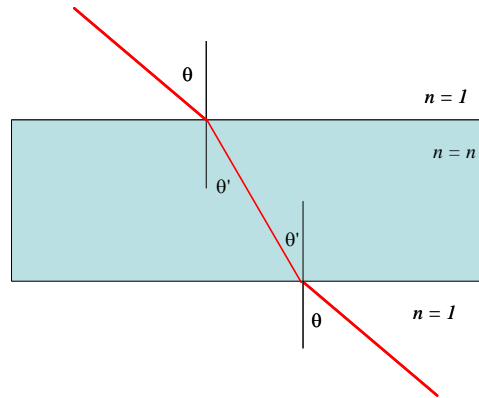


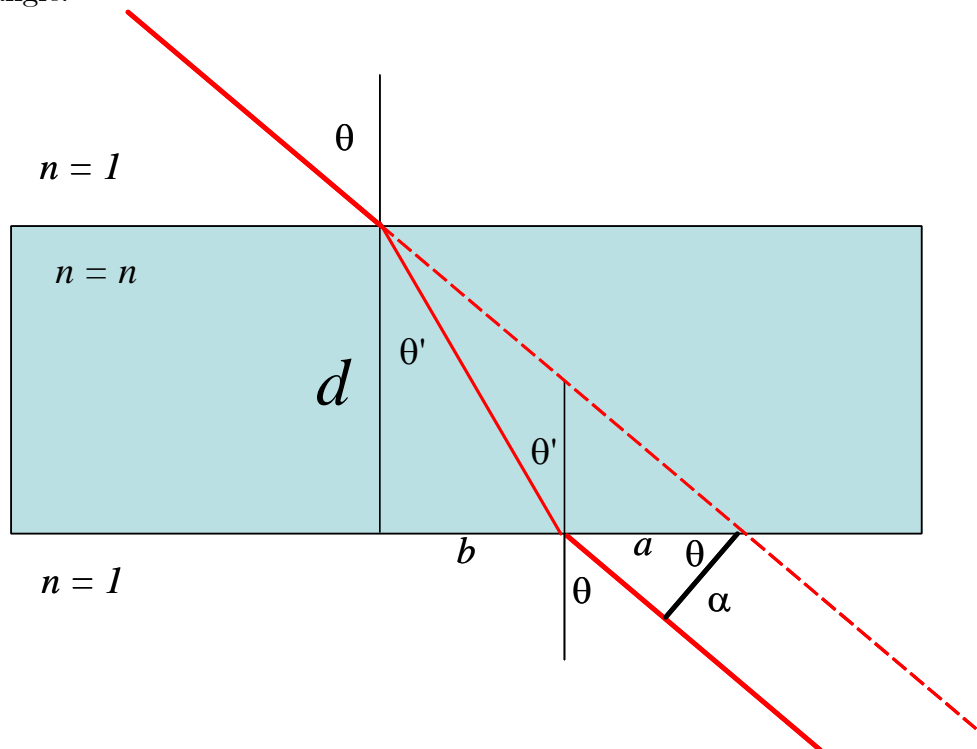
SIMG-712-01-20042 Solution Set #4

1. Consider a planar glass plate (i.e., the faces of the plate are parallel) of index n immersed in air
 - (a) Show that a ray incident at angle θ to the surface will emerge from the plate at the same angle.



A ray of light incident at angle θ on a plane-parallel plate of glass will be refracted at angle θ' obeying Snell's law. Since the sides are parallel, that ray will encounter the second surface at angle θ' and be refracted at angle θ by Snell's law.

- (b) If the thickness of the plate is d units, derive an expression for the physical displacement a of the emerging ray relative to the original ray as a function of the incident angle.



From the drawing, we can see two reasonable distances that would be defined at

a ; I have labeled them “ a ” and “ α ”– I would choose the latter (the displacement perpendicular to the path). From the drawing, we have that:

$$\begin{aligned} a + b &= d \tan [\theta] \\ b &= d \tan [\theta'] \\ \implies a &= d \tan [\theta] - d \tan [\theta'] \\ &= d \left(\frac{\sin [\theta]}{\cos [\theta]} - \frac{\sin [\theta']}{\cos [\theta']} \right) = d \left(\frac{\sin [\theta]}{\sqrt{1 - \sin^2 [\theta]}} - \frac{\sin [\theta']}{\sqrt{1 - \sin^2 [\theta']}} \right) \end{aligned}$$

Snell’s law tells us that:

$$\begin{aligned} n_1 \sin [\theta_1] &= n_2 \sin [\theta_2] \\ \implies \sin [\theta'] &= \frac{1}{n} \sin [\theta] \end{aligned}$$

So that we can evaluate a in terms of n and θ :

$$\begin{aligned} a &= d \left(\frac{\sin [\theta]}{\sqrt{1 - \sin^2 [\theta]}} - \frac{\frac{1}{n} \sin [\theta]}{\sqrt{1 - \left(\frac{1}{n}\right)^2 \sin^2 [\theta]}} \right) \\ &\boxed{a = d \sin [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{n^2 - \sin^2 [\theta]}} \right)} \end{aligned}$$

From the drawing:

$$\begin{aligned} \alpha &= a \cos [\theta] = d \sin [\theta] \cos [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{n^2 - \sin^2 [\theta]}} \right) \\ &= d \sin [\theta] \left(\frac{\cos [\theta]}{\cos [\theta]} - \frac{\cos [\theta]}{\sqrt{n^2 - \sin^2 [\theta]}} \right) \\ &\boxed{\alpha = d \sin [\theta] \left(1 - \sqrt{\frac{1 - \sin^2 [\theta]}{n^2 - \sin^2 [\theta]}} \right)} \end{aligned}$$

But it isn’t good enough to just derive the expression, we need to test it to see if it gives reasonable results. For example, if $n = 1$, then the deviation should be 0 for all θ :

$$\begin{aligned} a(n = 1) &= d \sin [\theta] \left(\frac{1}{\sqrt{1 - \sin^2 [\theta]}} - \frac{1}{\sqrt{1^2 - \sin^2 [\theta]}} \right) = 0 \\ \alpha(n = 1) &= d \sin [\theta] \left(1 - \sqrt{\frac{1 - \sin^2 [\theta]}{1 - \sin^2 [\theta]}} \right) = 0 \end{aligned}$$

So these check out. What if $n \neq 0$? In that case, if $\theta = 0$ the displacement should still be 0:

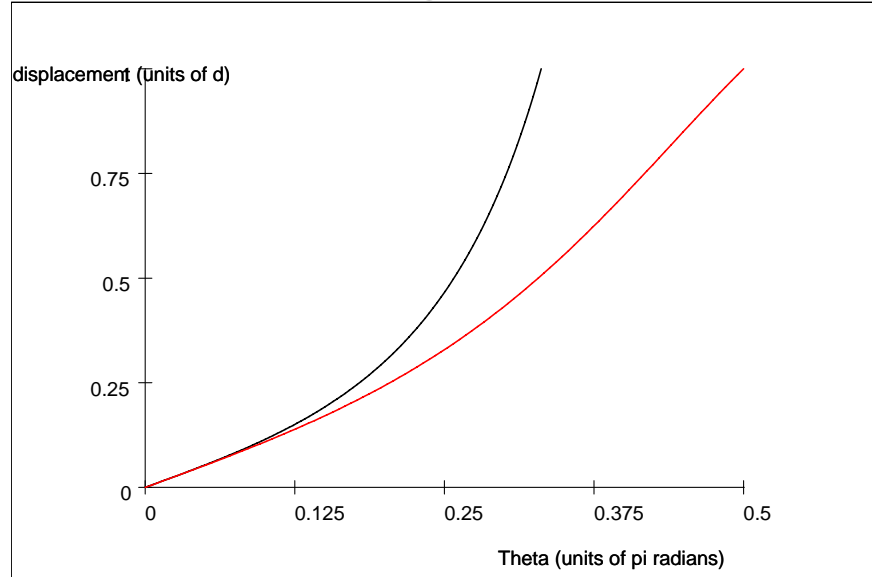
$$\begin{aligned} a[\theta = 0] &= a = d \sin [0] \left(\frac{1}{\sqrt{1 - \sin^2 [0]}} - \frac{1}{\sqrt{n^2 - \sin^2 [0]}} \right) = 0 \\ \alpha &= d \sin [0] \left(1 - \sqrt{\frac{1 - \sin^2 [0]}{n^2 - \sin^2 [0]}} \right) = 0 \end{aligned}$$

So that checks out too. How about if $\theta \neq 0$ for $n \neq 0$? Say $\theta = 45^\circ$

$$a \left[\theta = \frac{\pi}{4} \right] = d \sin \left[\frac{\pi}{4} \right] \left(\frac{1}{\sqrt{1 - \sin^2 \left[\frac{\pi}{4} \right]}} - \frac{1}{\sqrt{1.5^2 - \sin^2 \left[\frac{\pi}{4} \right]}} \right) \cong 0.465d < \frac{d}{2}$$

$$\alpha = d \sin \left[\frac{\pi}{4} \right] \left(1 - \sqrt{\frac{1 - \sin^2 \left[\frac{\pi}{4} \right]}{1.5^2 - \sin^2 \left[\frac{\pi}{4} \right]}} \right) \cong 0.329d < a < \frac{d}{2}$$

These seem to make sense. Just for fun, graph them:



Displacement a (black) and α (red) in units of the plate thickness d for $0 \leq \theta \leq \frac{\pi}{2}$.

2. A thin lens is made of glass with index $n = 1.53$. In air, the lens has a focal length $f = 254 \text{ mm}$. What is its focal length when it is totally immersed in water ($n = 1.33$)?

$$\begin{aligned}
 \text{Lensmaker's equation} & : \quad \frac{n_2}{f} = (n_2 - n_1) \cdot \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\
 \text{If in air:} & \quad \frac{1}{f} = \frac{1}{254 \text{ mm}} = \frac{(1.53 - 1.0)}{1.0} \cdot \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\
 & = 0.53 \cdot \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\
 \Rightarrow & \quad \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{0.53 \cdot 254 \text{ mm}} = \frac{1}{134.62 \text{ mm}} \\
 \text{If in water:} & \quad \frac{1}{f} = \frac{(1.53 - 1.33)}{1.33} \cdot \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \\
 f & = \left(\frac{0.20}{1.33} \cdot \frac{1}{0.53 \cdot 254 \text{ mm}} \right)^{-1} \Rightarrow \boxed{f_{\text{water}} \cong 895.2 \text{ mm}}
 \end{aligned}$$

The focal length of the lens in water is considerably longer than its focal length in air because the “relative refractivity” of the glass is much reduced in water. To see that this result makes sense, consider what the focal length of the lens would be if immersed in glass. The “relative refractivity” of the lens vanishes, so the focal length becomes infinite.

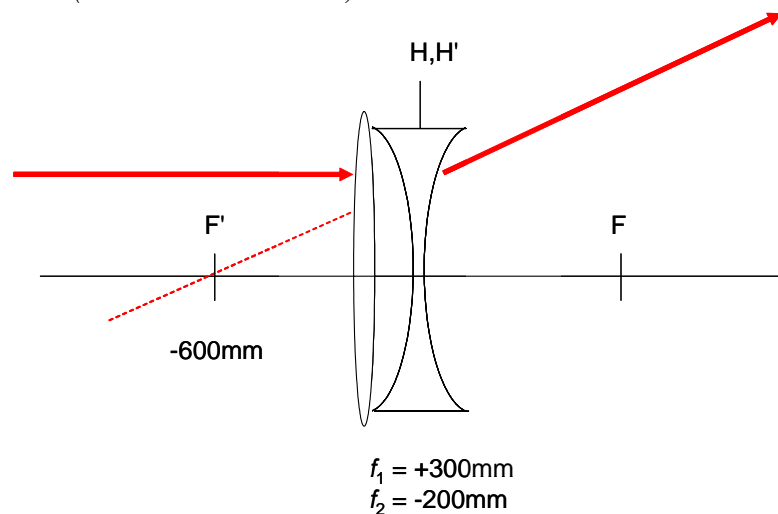
3. A convex thin lens with focal length $f_1 = 300$ mm and a concave thin lens with focal length $f_2 = -200$ mm are placed in contact.

- (a) Determine the focal length of the system.

$$\begin{aligned} \frac{1}{f_{eff}} &= \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \\ \Rightarrow \frac{1}{f_{eff}} &= \frac{1}{f_1} + \frac{1}{f_2} - 0 \\ f_{eff} &= \left(\frac{1}{300 \text{ mm}} + \frac{1}{-200 \text{ mm}} \right)^{-1} = -600 \text{ mm} \end{aligned}$$

- (b) Locate the principal points and focal points.

Since the two thin lenses are in contact, the principal points coincide with the location of the lenses (which also coincide).



- (c) Characterize the image formed by this system of an object located $s_1 = 400$ mm away.

$$\begin{aligned} \frac{1}{f_{eff}} &= \frac{1}{s_1} + \frac{1}{s_2} \\ \Rightarrow s_2 &= \left(\frac{1}{f_{eff}} - \frac{1}{s_1} \right)^{-1} = \left(\frac{1}{-600 \text{ mm}} - \frac{1}{400 \text{ mm}} \right)^{-1} = -240 \text{ mm} \\ M_T &= -\frac{s_2}{s_1} = -\frac{-240 \text{ mm}}{400 \text{ mm}} = +\frac{3}{5}; \text{ upright and minified} \end{aligned}$$

4. A system consists of two thin lenses L_1 ($f_1 = -60$ mm) and L_2 ($f_2 = ?$) separated by $t = 120$ mm. Lens L_2 is made of glass with $n = 1.5$ and is plano-convex; the radius of the curved side is $R = 60$ mm.

- (a) Locate and describe the image of an object that is 5 mm high located 180 mm "in front" of the first lens.

Lensmaker's equation:

$$\frac{1}{f_2} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$f_2 = \left(0.5 \left(\frac{1}{\infty} - \frac{1}{-60 \text{ mm}} \right) \right)^{-1} = +120 \text{ mm}$$

Brute-force calculation:

$$\frac{1}{s_1} + \frac{1}{s'_1} = \frac{1}{f_1} \implies s'_1 = \left(\frac{1}{-60 \text{ mm}} - \frac{1}{+180 \text{ mm}} \right)^{-1} = -45 \text{ mm}$$

$$M_{T1} = -\frac{s'_1}{s_1} = -\frac{-45 \text{ mm}}{+180 \text{ mm}} = +\frac{1}{4}$$

$$s_2 = t - s'_1 = 120 \text{ mm} - (-45 \text{ mm}) = +165 \text{ mm}$$

$$\frac{1}{s_2} + \frac{1}{s'_2} = \frac{1}{f_2} \implies s'_2 = \left(\frac{1}{+120 \text{ mm}} - \frac{1}{+165 \text{ mm}} \right)^{-1} = +440 \text{ mm}$$

$$M_{T2} = -\frac{s'_2}{s_2} = -\frac{+440 \text{ mm}}{+165 \text{ mm}} = -\frac{8}{3}$$

$$M_T = M_{T1} \cdot M_{T2} = \left(+\frac{1}{4} \right) \cdot \left(-\frac{8}{3} \right) = -\frac{2}{3}$$

$$h' = h \cdot M_T = 5 \text{ mm} \cdot -\frac{2}{3} = -\frac{10}{3} \text{ mm} \cong -3.33 \text{ mm}$$

- (b) Determine the focal length of the system and find the focal points and principal points.

$$\frac{1}{f_{eff}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}$$

$$f_{eff} = \left(\frac{1}{-60 \text{ mm}} + \frac{1}{+120 \text{ mm}} - \frac{120 \text{ mm}}{(-60 \text{ mm})(+120 \text{ mm})} \right)^{-1} = \boxed{+120 \text{ mm} = f_{eff}}$$

note that the focal length of the system is the same as that of f_2

because the first lens is located at the front focal point of the second lens....

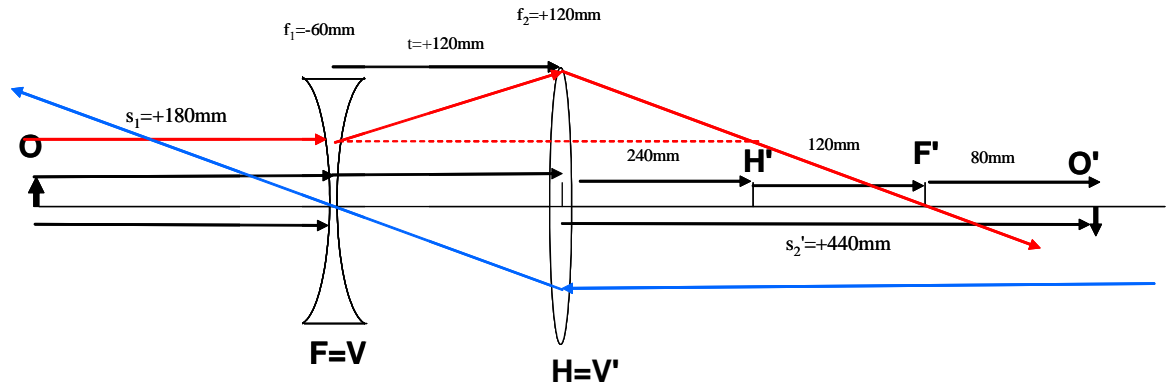
Locate the image-space focal point of the system:

$$s'_1 = f_1 = -60 \text{ mm}$$

$$s_2 = t - s'_1 = 120 \text{ mm} - (-60 \text{ mm}) = +180 \text{ mm}$$

$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{120 \text{ mm}} - \frac{1}{180 \text{ mm}} \right)^{-1} = \boxed{+360 \text{ mm} = \overline{\mathbf{V}'\mathbf{F}'}}$$

$$\overline{\mathbf{V}'\mathbf{H}'} + \overline{\mathbf{H}'\mathbf{F}'} = \overline{\mathbf{V}'\mathbf{F}'} \implies \overline{\mathbf{V}'\mathbf{H}'} = \overline{\mathbf{V}'\mathbf{F}'} - \overline{\mathbf{H}'\mathbf{F}'} = +360 \text{ mm} - 240 \text{ mm} = \boxed{\overline{\mathbf{V}'\mathbf{H}'} = 120 \text{ mm}}$$



Reverse it to find the object-space focal point:

$$\begin{aligned}
 s_2' &= \infty \implies s_2 = f_2 = +120 \text{ mm} \\
 s_1 &= t - s_2' = 120 \text{ mm} - 120 \text{ mm} = 0 \text{ mm} \\
 s_1' &= 0 \text{ mm} \implies \overline{FV} = 0 \text{ mm} \implies \text{object-space focal point at } \mathbf{V} \\
 &\implies \mathbf{H} \text{ located } 120 \text{ mm to the right of } \mathbf{V}, \text{ i.e., at } \mathbf{V}'
 \end{aligned}$$

Test it by finding image of the object with $\overline{OV} = 180 \text{ mm}$, then:

$$\begin{aligned}
 \overline{OH} &= 180 \text{ mm} + 120 \text{ mm} = 300 \text{ mm} = s \\
 s' &= \overline{H'O'} = \left(\frac{1}{f_{eff}} - \frac{1}{s} \right)^{-1} = \left(\frac{1}{120 \text{ mm}} - \frac{1}{300 \text{ mm}} \right)^{-1} = +200 \text{ mm} \\
 \overline{V'O'} &= \overline{V'H'} + \overline{H'O'} = 240 \text{ mm} + 200 \text{ mm} = +440 \text{ mm} \\
 M_T &= -\frac{s'}{s} = -\frac{200 \text{ mm}}{300 \text{ mm}} = -\frac{2}{3}
 \end{aligned}$$

5. Two thin lenses having focal lengths $f_1 = +150$ mm and $f_2 = -150$ mm are separated by $t = 600$ mm.

(a) A page of print is held 250 mm in front of the positive lens. For paraxial rays, determine the location, orientation, and magnification of the image of the print.

$$\begin{aligned}
 s'_1 &= \left(\frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left(\frac{1}{+150 \text{ mm}} - \frac{1}{250 \text{ mm}} \right)^{-1} = 375 \text{ mm} \\
 (M_T)_1 &= -\frac{s'_1}{s_1} = -\frac{375 \text{ mm}}{250 \text{ mm}} = -\frac{3}{2} \\
 s_2 &= t - s'_1 = 600 \text{ mm} - 375 \text{ mm} = 225 \text{ mm} \\
 s'_2 &= \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-150 \text{ mm}} - \frac{1}{225 \text{ mm}} \right)^{-1} = -90 \text{ mm} \implies \text{virtual image} \\
 (M_T)_2 &= -\frac{s'_2}{s_2} = -\frac{-90 \text{ mm}}{225 \text{ mm}} = +\frac{2}{5} \\
 M_T &= (M_T)_1 (M_T)_2 = -\frac{3}{2} \cdot +\frac{2}{5} = -\frac{3}{5}, \text{ inverted image}
 \end{aligned}$$

(b) Determine the focal length of the system and find the focal points and principal points.

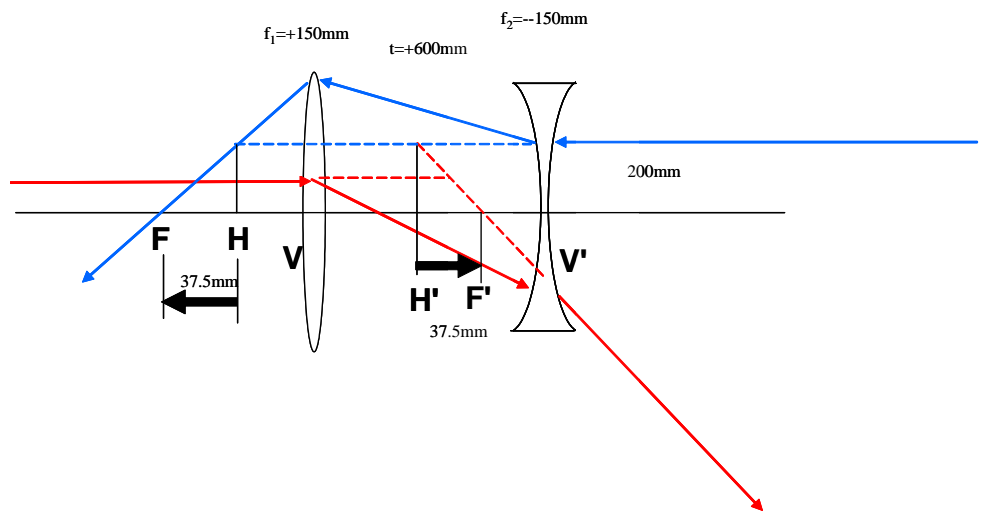
$$\begin{aligned}
 f_{eff} &= \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \right)^{-1} \\
 &\boxed{f_{eff} = \frac{75}{2} \text{ mm}}
 \end{aligned}$$

Find \mathbf{F}' , place object at ∞

$$\begin{aligned}
 s'_1 &= \left(\frac{1}{f_1} - \frac{1}{\infty} \right)^{-1} = 150 \text{ mm} \\
 s_2 &= t - s'_1 = 600 \text{ mm} - 150 \text{ mm} = 450 \text{ mm} \\
 s'_2 &= \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-150 \text{ mm}} - \frac{1}{450 \text{ mm}} \right)^{-1} = -\frac{225}{2} \text{ mm} = \overline{\mathbf{V}'\mathbf{F}'} \\
 \overline{\mathbf{H}'\mathbf{F}'} &= f_{eff} = \frac{75}{2} \text{ mm} \\
 \overline{\mathbf{V}'\mathbf{H}'} &= \overline{\mathbf{V}'\mathbf{F}'} - \overline{\mathbf{H}'\mathbf{F}'} = -\frac{225}{2} \text{ mm} - \frac{75}{2} \text{ mm} = \boxed{\overline{\mathbf{V}'\mathbf{H}'} = -150 \text{ mm}}
 \end{aligned}$$

Find \mathbf{F} , place image at ∞

$$\begin{aligned}
 s_2 &= \left(\frac{1}{f_2} - \frac{1}{s'_1} \right)^{-1} = \left(\frac{1}{-150 \text{ mm}} - \frac{1}{\infty} \right)^{-1} = -150 \text{ mm} \\
 s'_1 &= t - s_2 = 600 \text{ mm} - (-150 \text{ mm}) = 750 \text{ mm} \\
 s_1 &= \left(\frac{1}{f_1} - \frac{1}{s'_1} \right)^{-1} = \left(\frac{1}{150 \text{ mm}} - \frac{1}{750 \text{ mm}} \right)^{-1} = \frac{375}{2} \text{ mm} = \overline{\mathbf{F}\mathbf{V}} \\
 \overline{\mathbf{V}\mathbf{H}} &= \overline{\mathbf{F}\mathbf{H}} - \overline{\mathbf{F}\mathbf{V}} = f_{eff} - \overline{\mathbf{F}\mathbf{V}} = \frac{75}{2} \text{ mm} - \frac{375}{2} \text{ mm} = -150 \text{ mm} \implies \boxed{\overline{\mathbf{H}\mathbf{V}} = +150 \text{ mm}}
 \end{aligned}$$



6. The images generated by a biconvex lens in air from objects at $s_1 = \infty$ and $s_1 = 200$ mm are located 8 mm apart. Determine the focal length of the lens.

$$\begin{aligned}
 \frac{1}{s_1} + \frac{1}{s'_1} &= \frac{1}{f} = \frac{1}{s_2} + \frac{1}{s'_2} \\
 \frac{1}{f} &= \frac{1}{\infty} + \frac{1}{f} = \frac{1}{200 \text{ mm}} + \frac{1}{f + 8 \text{ mm}} \\
 &\implies \frac{1}{200 \text{ mm}} = \frac{1}{f} - \frac{1}{f + 8 \text{ mm}} \\
 200 \text{ mm} &= \frac{f^2 + f \cdot 8 \text{ mm}}{8 \text{ mm}} \\
 f^2 + f \cdot 8 \text{ mm} - 1600 \text{ mm}^2 &= 0 \\
 f &= \frac{-8}{2} \text{ mm} \pm \frac{\sqrt{(8 \text{ mm})^2 + 4 \cdot 1600 \text{ mm}^2}}{2} \\
 &\boxed{f_1 \cong +36.2 \text{ mm}, f_2 \cong -44.2 \text{ mm}}
 \end{aligned}$$

So we can get this result for either a positive or negative lens. Check it:

$$\begin{aligned}
 f_1 \cong +36.2 \text{ mm}, s = \infty &\implies s' = f_1 \cong +36.2 \text{ mm} \\
 s_1 = +200 \text{ mm} &\implies s'_1 = \left(\frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left(\frac{1}{36.2 \text{ mm}} - \frac{1}{200 \text{ mm}} \right)^{-1} = +44.2 \text{ mm} \\
 |s' - s'_1| &\cong |+36.2 \text{ mm} - 44.2 \text{ mm}| = 8 \text{ mm}
 \end{aligned}$$

$$\begin{aligned}
 f_2 \cong -44.2 \text{ mm}, s = \infty &\implies s' = f_2 \cong -44.2 \text{ mm} \\
 s_2 = +200 \text{ mm} &\implies s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-44.2 \text{ mm}} - \frac{1}{200 \text{ mm}} \right)^{-1} = -36.2 \text{ mm} \\
 |s' - s'_1| &\cong |(-44.2 \text{ mm}) - (-36.2 \text{ mm})| = 8 \text{ mm}
 \end{aligned}$$

So the constraint is satisfied for both values of the focal length.

7. Three lenses with focal lengths $f_1 = +100$ mm, $f_2 = -100$ mm, and $f_3 = +100$ mm are placed in that order and each is separated from the next by $t_n = 20$ mm.

(a) Determine the focal length of the system.

Find the equivalent single lens for the first two lenses:

$$\begin{aligned} f_{12} &= \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \right)^{-1} \\ &= \left(\frac{1}{+100 \text{ mm}} + \frac{1}{(-100 \text{ mm})} - \frac{20 \text{ mm}}{(+100 \text{ mm})(-100 \text{ mm})} \right)^{-1} = +500 \text{ mm} \end{aligned}$$

\mathbf{F}'_{12} = image position from two lenses for object at ∞

$$\frac{1}{s'_1} = \frac{1}{f_1} \implies s'_1 = +100 \text{ mm} \implies s_2 = t_{12} - s'_1 = 20 \text{ mm} - 100 \text{ mm} = -80 \text{ mm}$$

$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-100 \text{ mm}} - \frac{1}{-80 \text{ mm}} \right)^{-1} = 400 \text{ mm}$$

Because $f_{12} = +500$ mm, \mathbf{H}_{12} is located 500 mm “in front” of \mathbf{F}'_{12} ,
or 100 mm “in front” of \mathbf{V}_2

Now combine the equivalent thin lens for 12 with 3

$$t_{23} = 100 \text{ mm} + 20 \text{ mm} = 120 \text{ mm}$$

$$\begin{aligned} f_{eff} &= \left(\frac{1}{f_{12}} + \frac{1}{f_3} - \frac{t_{23}}{f_{12} f_3} \right)^{-1} \\ &= \left(\frac{1}{+500 \text{ mm}} + \frac{1}{+100 \text{ mm}} - \frac{120 \text{ mm}}{(+500 \text{ mm})(+100 \text{ mm})} \right)^{-1} = \frac{625}{6} \text{ mm} \end{aligned}$$

$$\boxed{f_{eff} = \frac{625}{6} \text{ mm} = 104\frac{1}{6} \text{ mm}}$$

(b) Locate the principal and focal points.

For image-space focal point, bring in a ray from the object side parallel to the optical axis:

$$s'_1 = f_1 = +100 \text{ mm}$$

$$s_2 = t_1 - s'_1 = 20 \text{ mm} - 100 \text{ mm} = -80 \text{ mm}$$

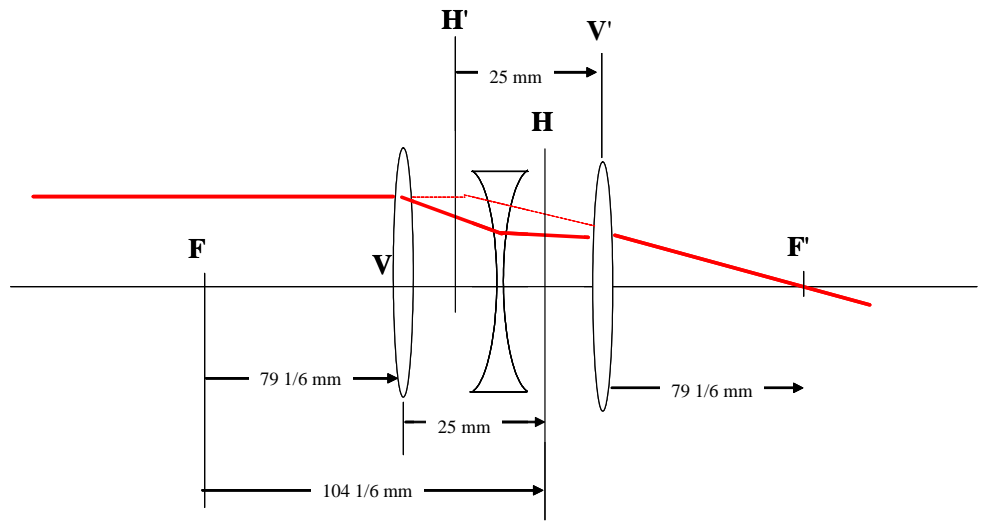
$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{(-100 \text{ mm})} - \frac{1}{(-80 \text{ mm})} \right)^{-1} = 400 \text{ mm}$$

$$s_3 = t_2 - s'_2 = 20 \text{ mm} - 400 \text{ mm} = -380 \text{ mm}$$

$$s'_3 = \left(\frac{1}{f_3} - \frac{1}{s_3} \right)^{-1} = \left(\frac{1}{100 \text{ mm}} - \frac{1}{-380 \text{ mm}} \right)^{-1} = \frac{475}{6} \text{ mm} = 79\frac{1}{6} \text{ mm} = \overline{\mathbf{V}'\mathbf{F}'}$$

$$\overline{\mathbf{H}'\mathbf{F}'} = f_{eff} = 104\frac{1}{6} \text{ mm} \implies \overline{\mathbf{H}'\mathbf{V}'} = \overline{\mathbf{H}'\mathbf{F}'} - \overline{\mathbf{V}'\mathbf{F}'} = 104\frac{1}{6} \text{ mm} - 79\frac{1}{6} \text{ mm} = 25 \text{ mm}$$

Because system is completely symmetric, the object -space principal and focal points are symmetrically placed.... :



8. A gypsy has a crystal ball, of index of refraction $n = 1.6$ and diameter 200 mm that may be used as a lens. Locate the principal and focal points.

$$\begin{aligned} R_1 &= +100 \text{ mm} \\ R_2 &= -100 \text{ mm} \\ t &= +200 \text{ mm} \\ \frac{t}{n} &= \frac{+200 \text{ mm}}{1.6} = +125 \text{ mm} \end{aligned}$$

Trace a provisional marginal ray. Ray at unit height is refracted by the first surface: $\varphi_1 = \frac{n-1}{R_1} = \frac{0.6}{+100 \text{ mm}} = \frac{0.6}{0.1 \text{ m}} = 6 \text{ diopters} = 0.006 \text{ mm}^{-1}$ Angle after first surface:

$$\begin{aligned} n'u' &= nu - y_1\varphi_1 = -y_1\varphi_1 \text{ because } u = 0 \text{ and } y_1 = 1 \\ n'u' &= -\varphi_1 = -0.006 \text{ radians} \end{aligned}$$

Transfer to second surface:

$$\begin{aligned} y' &= y + \frac{t'}{n'}n'u' \\ &= 1 + \frac{200 \text{ mm}}{1.6} (1.6 \cdot -0.006 \text{ mm}^{-1}) = -0.2 \\ &1 + \frac{200 \text{ mm}}{1.6} (-0.006 \text{ mm}^{-1}) = 0.25 \\ \varphi_2 &= \frac{1-n}{R_2} = \frac{-0.6}{-100 \text{ mm}} = +0.006 \text{ mm}^{-1} \end{aligned}$$

Refraction at second surface:

$$n'u' = -0.006 \text{ radians} - (0.25 \text{ mm}) (+0.006 \text{ mm}^{-1}) = -0.0075 \text{ radians}$$

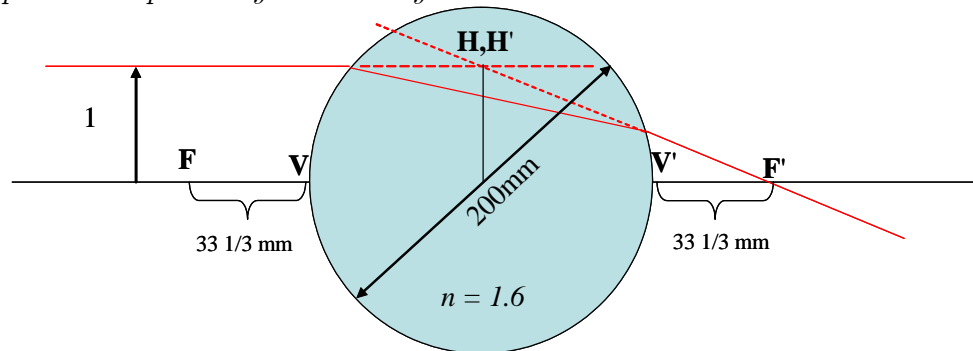
height is 0.25, angle is -0.0075 , crosses axis at:

$$t' = -\frac{+0.25}{-0.0075} = 33\frac{1}{3} \text{ mm}$$

principal point is located where exiting ray with angle -0.0075 radians has a height of one (height of incoming ray)

$$\overline{H'F'} = f_{eff} = -\frac{1 \text{ mm}}{-0.0075} = +\frac{400}{3} \text{ mm} = 133\frac{1}{3} \text{ mm}$$

which is at the center of the sphere. Since the system is symmetric, both focal and principal points are placed symmetrically.



9. The surfaces of a thin equiconvex lens have equal radius of curvature: $|R_1| = 150$ mm. The second surface is aluminized to reflect light. Find the location of the image of an object located 400 mm to the left of the first surface.

I think of this system as three thin “lenses” separated by $t = 0$.

The power of the first lens is

$$\varphi_1 = \frac{1}{f_1} = \frac{(n_2 - n_1)}{R_1}$$

Since the lens is glass, assume that $n_2 = 1.5$

$$\varphi_1 = \frac{0.5}{+150 \text{ mm}} = \frac{1}{+300 \text{ mm}}$$

The power of the second “lens” (the mirror) determines the angle of reflection of the outgoing ray. The indices of the incoming and outgoing spaces are both glass, and thus the ray angle does not depend on the index. The power of the mirror is:

$$\varphi_2 = \frac{1}{f_2} = \frac{(-1) - (+1)}{R_2} = -\frac{2}{-150 \text{ mm}} = +\frac{1}{75 \text{ mm}}$$

(where φ_2 is positive because the “lens” makes the rays converge to a focus rather than diverge from a focus)

The power of the third lens (the front surface again) is:

$$\varphi_3 = \frac{1}{f_3} = \frac{(n_1 - n_2)}{-R_1} \quad (\text{where the radius is negative because light is going from right to left})$$

$$\varphi_3 = \frac{-0.5}{-150 \text{ mm}} = \frac{1}{+300 \text{ mm}} \quad (\text{same as } \varphi_1; \text{ both make the rays converge})$$

The three “lenses” are in contact, so the power of the system is the sum of the powers:

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 = \frac{1}{300 \text{ mm}} + \frac{1}{75 \text{ mm}} + \frac{1}{300 \text{ mm}} = \frac{1}{50 \text{ mm}} \implies f = +50 \text{ mm}$$

object distance: $s = 400$ mm

object and image are in air so $n_1 = n_3 = 1.0$

$$\frac{1}{s} + \frac{1}{s'} = \varphi \implies s' = \left(\frac{1}{50 \text{ mm}} - \frac{1}{400 \text{ mm}} \right)^{-1} = \frac{400}{7} \text{ mm} \cong \boxed{51.14 \text{ mm} \cong s'}$$

$$M_T = -\frac{s'}{s} = -\frac{\frac{400}{7} \text{ mm}}{400 \text{ mm}} = -\frac{1}{7} \cong -0.14$$

*The image is approximately 51 mm **to the left** of the first surface (despite the fact that the sign is positive) because of the reflection. The image is real, inverted, and minified.*

Compare this result to the image created by the lens alone:

$$\varphi = \varphi_1 + \varphi_3 = \frac{1}{+300 \text{ mm}} + \frac{1}{+300 \text{ mm}} = \frac{2}{+300 \text{ mm}} = \frac{1}{+150 \text{ mm}} \implies f = 150 \text{ mm}$$

$$s' = \left(\varphi - \frac{1}{s} \right)^{-1} = \left(\frac{1}{+150 \text{ mm}} - \frac{1}{400 \text{ mm}} \right)^{-1} = +240 \text{ mm}, \quad M_T = -\frac{240 \text{ mm}}{400 \text{ mm}} = -\frac{3}{5} = -0.6$$

The focal length of the “lens-mirror” system is much shorter than of the lens alone.