SIMG-712-01-20042 Solution Set #2

1. Show that the following waveforms are solutions to the differential wave equation

$$\begin{aligned}\psi_a[z,t] &= A_0\sin[k_0(z-\mathrm{v}_0t)]\\ \psi_b[z,t] &= A_0\cos[k_0z-\omega_0t]\end{aligned}$$

(a)

$$\begin{aligned}
\frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} &= \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\\
\nabla^2(A_0\sin[k_0(z-\mathrm{v}_0t)]) &= A_0\nabla^2(\sin[k_0(z-\mathrm{v}_0t)])\\
&= A_0\left(0+0+\frac{\partial^2}{\partial z^2}\sin[k_0(z-\mathrm{v}_0t)]\right)\\
&= A_0\frac{\partial}{\partial z}(+k_0\cos[k_0(z-\mathrm{v}_0t)])\\
&= A_0\left((+k_0)^2\cdot-\sin[k_0(z-\mathrm{v}_0t)]\right)\\
&= -A_0k_0^2\sin[k_0(z-\mathrm{v}_0t)]\\
\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(A_0\sin[k_0(z-\mathrm{v}_0t)])\\
&= A_0\frac{\partial}{\partial t}(-(k_0\mathrm{v}_0)\cos[k_0(z-\mathrm{v}_0t)])\\
&= A_0(-k_0\mathrm{v}_0)^2\cdot-\sin[k_0(z-\mathrm{v}_0t)]\\
&= -A_0\left(+k_0^2\mathrm{v}_0^2\right)\sin[k_0(z-\mathrm{v}_0t)]\\
&\Longrightarrow \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = -A_0k_0^2\sin[k_0(z-\mathrm{v}_0t)] = \nabla^2\psi
\end{aligned}$$

(b)

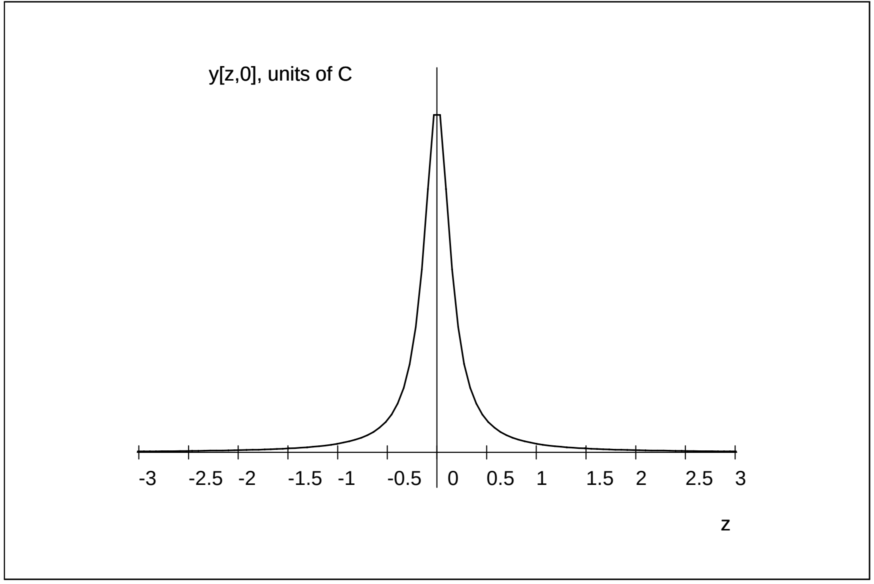
$$\begin{aligned}
\nabla^2(A_0\cos[k_0z-\omega_0t]) &= A_0\nabla^2(\cos[k_0z-\omega_0t])\\
&= A_0\left(0+0+\frac{\partial^2}{\partial z^2}\cos[k_0z-\omega_0t]\right)\\
&= A_0\frac{\partial}{\partial z}(-k_0\sin[k_0z-\omega_0t])\\
&= A_0((-k_0\cdot+k_0)\cdot+\cos[k_0z-\omega_0t])\\
&= -A_0k_0^2\cos[k_0z-\omega_0t]\\
\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial^2}{\partial t^2}(A_0\cos[k_0z-\omega_0t])\\
&= A_0\frac{\partial}{\partial t}(-\omega_0\cdot-\sin[k_0z-\omega_0t])\\
&= -A_0(-\omega_0)^2\cdot+\cos[k_0z-\omega_0t]\\
&= -A_0\omega_0^2\cos[k_0z-\omega_0t]\\
&\Longrightarrow \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = -A_0\left(\frac{k_0^2}{\omega_0^2}\right)\cdot\omega_0^2\cos[k_0z-\omega_0t]\\
&= -A_0k_0^2\cos[k_0z-\omega_0t] = \nabla^2\psi
\end{aligned}$$

2. A waveform has the shape:

$$y\left[z, t=0\right] = \frac{C}{1+(2\pi z)^2}$$

where C is some numerical constant.

(a) Sketch the profile of the wave $y\left[z,0\right]$.

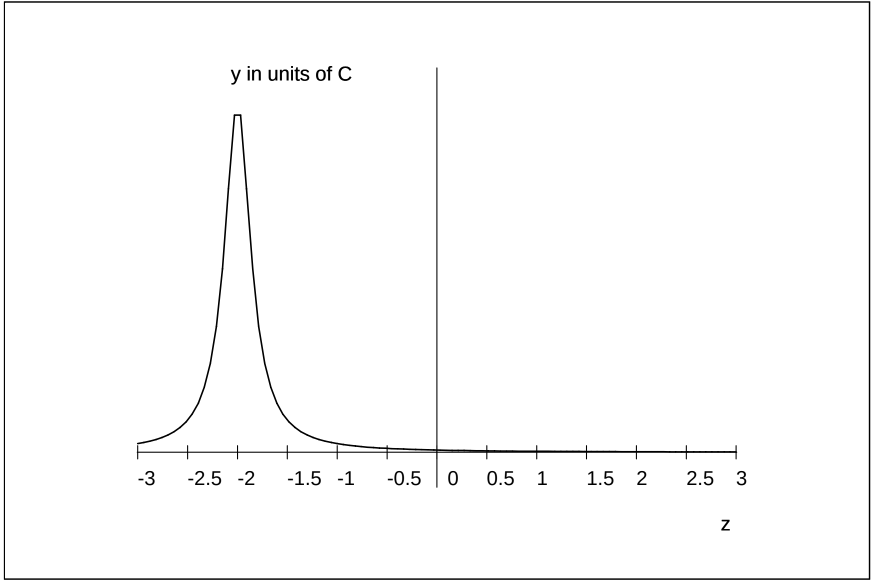


$y\left[z,0\right] = \frac{C}{1+(2\pi z)^2}$ *for* $C = 1$

(b) Write an expression for the wave as a function of z and t if it travels with velocity v_0 towards $z = -\infty$.

$$y\left[z,t\right] = \frac{C}{1+\left(2\pi\left(z+\mathrm{v}_0 t\right)\right)^2}$$

(c) Assume that v $= 1\,\frac{\mathrm{m}}{\mathrm{s}}$. Sketch the profile $y\left[z, t = 2\,\mathrm{s}\right]$



$y\left[z, t=2\,\mathrm{s}\right] = \frac{C}{1+\left[2\pi\left(z+\mathrm{v}_0\cdot 2\,\mathrm{s}\right)\right]^2}$

3. An isotropic monochromatic point source radiates at angular temporal frequency of ω_0 in vacuum with a power of 100 W.

   (a) What is the flux density at a distance of 1 m?

   Since this is an isotropic point source, it emits the same radiation into each unit of solid angle. At a radius of 1 m, *the area of the spherical shell enclosing the source is:*

$$A = 4\pi r^2 = 4\pi\,\mathrm{m}^2 \cong 12.56\,\mathrm{m}^2$$

   The power flux density through this surface is the specified 100 W, *so the power flux density is:*

$$I = \frac{100\,\mathrm{W}}{4\pi\,\mathrm{m}^2} \cong 7.96\frac{\mathrm{W}}{\mathrm{m}^2}$$

   (b) What are the amplitudes of the **E-** and **B**-fields at that distance?

   The electric field is measured in volts per meter; the magnetic field in volt-seconds per squre meter (the "tesla"). Use the Poynting vector for energy flow:

$$\underline{\mathbf{s}} = c^2\epsilon_0\underline{\mathbf{E}}\times\underline{\mathbf{B}}$$

   The time average of the magnitude of the Poynting vector is the "average energy per unit area per unit time."

$$\begin{aligned} I &= \langle|\underline{\mathbf{s}}|\rangle = c^2\epsilon_0\cdot\frac{1}{2}\cdot|\underline{\mathbf{E}}\times\underline{\mathbf{B}}| \\ &= \frac{c\epsilon_0}{2}E_0^2 \end{aligned}$$

   because the average value of the square of the cosine is $\frac{1}{2}$.

$$\begin{aligned} E_0^2 &= \frac{2I}{c\epsilon_0} = \frac{2\cdot\frac{100\,\mathrm{W}}{4\pi\,\mathrm{m}^2}}{(3\times10^8\,\mathrm{m\,s^{-1}})\cdot(8.85\times10^{-12}\,\mathrm{F\,m^{-1}})} \\ &\cong 5996\frac{\mathrm{W-s}}{\mathrm{F-m^2}} = 5996\left(\frac{\mathrm{kg-m}}{\mathrm{s^2}}\times\frac{1}{\mathrm{A-s}}\right)^2 = 5996\left(\frac{\mathrm{N}}{\mathrm{C}}\right)^2 = 5996\left(\frac{\mathrm{V}}{\mathrm{m}}\right) \\ E_0 &\cong \sqrt{5996}\frac{\mathrm{V}}{\mathrm{m}} \cong 77.4\frac{\mathrm{V}}{\mathrm{m}} \\ B_0 &= \frac{E_0}{c} \cong \frac{77.4\frac{\mathrm{V}}{\mathrm{m}}}{3\times10^8\,\mathrm{m\,s^{-1}}} \cong 2.58\times10^{-7}\frac{\mathrm{V}}{\mathrm{m}}\frac{\mathrm{s}}{\mathrm{m}} = 2.58\times10^{-7}\frac{\mathrm{N}}{\mathrm{C}}\frac{\mathrm{s}}{\mathrm{m}} \\ &= 2.58\times10^{-7}\frac{\mathrm{N}}{\mathrm{A-m}} = 2.58\times10^{-7}\,\mathrm{T} \end{aligned}$$

   (c) Repeat for a distance of 2 m.

$$\begin{aligned} A &= 4\pi r^2 = 16\pi\,\mathrm{m}^2 \cong 50.265\,\mathrm{m}^2 \\ \textit{Flux density} &= I = \frac{100\,\mathrm{W}}{16\pi\,\mathrm{m}^2} \cong 1.99\frac{\mathrm{W}}{\mathrm{m}^2} \\ E_0 &= \frac{2I}{c\epsilon_0} \cong 38.7\frac{\mathrm{V}}{\mathrm{m}} \\ B_0 &\cong 1.29\times10^{-7}\,\mathrm{T} \end{aligned}$$

4. Consider cylindrical waves emitted by a line source

   (a) Make an argument from conservation of energy considerations that cylindrical waves must have an amplitude that decreases approximately as $\rho^{-\frac{1}{2}}$, where ρ is the radial coordinate in a cylindrical coordinate system and the waves are originating from a line at $\rho = 0$.

   The waves will be propagating radially, and therefore if we consider the Poynting vector as it passes through a cylindrical surface of any radius ρ, we must have conservation of energy; that is, the energy per unit time crossing the surface of the cylinder should be independent of ρ:

$$\frac{E}{t} = \int \langle \underline{\mathbf{s}} \rangle_t \cdot d\underline{\mathbf{a}} = \textit{constant } \alpha$$

   In this case, $\underline{\mathbf{s}}$ will be parallel to $d\underline{\mathbf{a}}$ everywhere except the caps of the cylinder, where it is perpendicular to $d\underline{\mathbf{a}}$. This means that:

$$2\pi\rho h \left\langle |\underline{\mathbf{s}}| \right\rangle_t = \alpha$$

   where h is the height of the cylinder in the z-direction. But, now considering the time-independent part of the waves,

$$|\underline{\mathbf{E}}| \propto \sqrt{|\underline{\mathbf{s}}|},$$

   so we have the required result:

$$|\underline{\mathbf{E}}| \propto \sqrt{\frac{\alpha}{2\pi\rho h}} \propto \frac{1}{\sqrt{\rho}}$$

   (b) Show explicitly that:

$$\psi(\rho, t) = \psi_o \frac{\exp\left[i(k_0\rho - \omega_0 t)\right]}{\sqrt{\rho}}$$

   is a solution to the three-dimensional wave equation,

$$\nabla^2\psi - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

   in cylindrical coordinates for large ρ.

   The Laplacian in cylindrical coordinates is:

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}$$

The last two terms vanish as $\frac{\exp[i(k_0\rho-\omega_0 t)]}{\sqrt{\rho}}$ *is independent of* φ *and* z*. Therefore,*

$$\begin{aligned}
\nabla^2\psi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left(\psi_0\frac{e^{i(k_0\rho-\omega_0 t)}}{\sqrt{\rho}}\right)\right) \\
&= \psi_0\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\left(\frac{\sqrt{\rho}ik_0e^{i(k_0\rho-\omega_0 t)}-\frac{1}{2}\rho^{-\frac{1}{2}}e^{i(k_0\rho-\omega_0 t)}}{\rho}\right)\right) \\
&= \psi_0\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\sqrt{\rho}ik_0e^{i(k_0\rho-\omega_0 t)}-\frac{1}{2}\rho^{-\frac{1}{2}}e^{i(k_0\rho-\omega_0 t)}\right) \\
&= \psi_0\frac{1}{\rho}\left(\sqrt{\rho}(ik_0)^2e^{i(k_0\rho-\omega_0 t)}+\frac{1}{2}\rho^{-\frac{1}{2}}ik_0e^{i(k_0\rho-\omega_0 t)}-\frac{1}{2}\rho^{-\frac{1}{2}}ik_0e^{i(k_0\rho-\omega_0 t)}+\frac{1}{4}\rho^{-\frac{3}{2}}e^{i(k_0\rho-\omega_0 t)}\right) \\
&= \psi_0\left(\frac{-k_0^2e^{i(k_0\rho-\omega_0 t)}}{\sqrt{\rho}}+\frac{1}{4\sqrt{\rho^5}}e^{i(k_0\rho-\omega_0 t)}\right)
\end{aligned}$$

At large ρ*, the first term in the parentheses will dominate because* $\rho^{-\frac{5}{2}}$ *falls off quickly, thus:*

$$\nabla^2\psi \approx -k_0^2\left(\psi_0\frac{e^{i(k_0\rho-\omega_0 t)}}{\sqrt{\rho}}\right) = -k_0^2\psi.$$

Now compute the time derivatives:

$$\begin{aligned}
\frac{\partial^2\psi}{\partial t^2} &= \psi_0\frac{\partial^2}{\partial t^2}\left(\frac{e^{i(k_0\rho-\omega_0 t)}}{\sqrt{\rho}}\right) \\
&= \psi_0\frac{-\omega_0^2e^{i(k_0\rho-\omega_0 t)}}{\sqrt{\rho}} = -\omega_0^2\psi
\end{aligned}$$

Combine the two sides:

$$\nabla^2\psi - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = -k_0^2\psi + \frac{\omega_0^2}{\mathrm{v}^2}\psi$$

Now, we know that by definition,

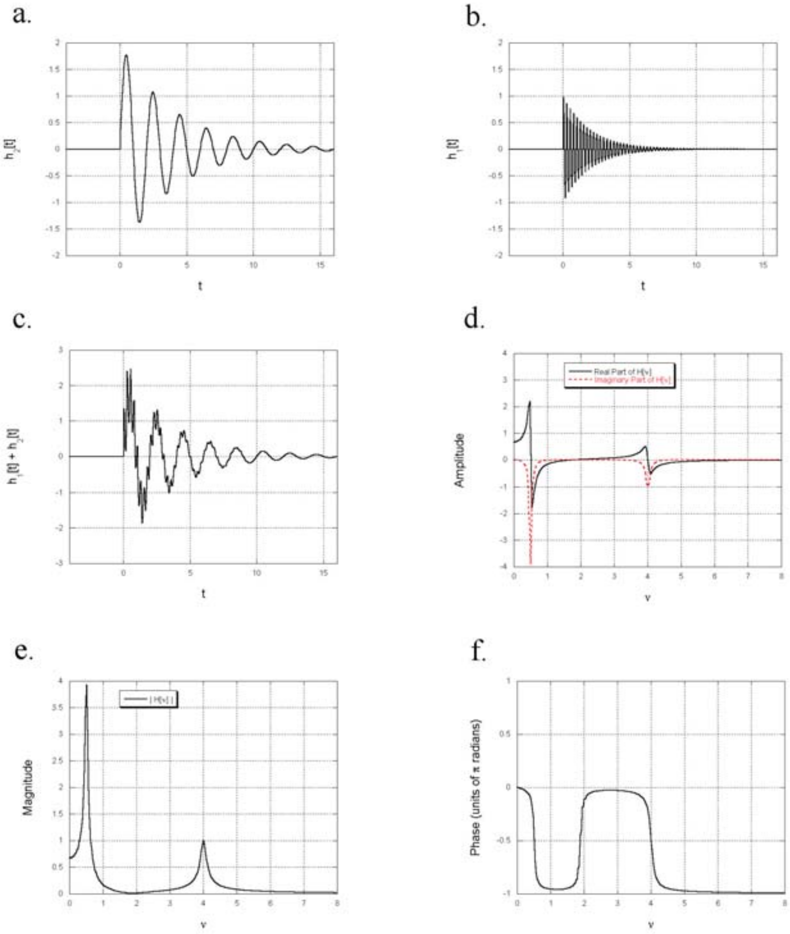
$$\mathrm{v} = \frac{\omega_0}{k_0} \implies \left(-k_0^2 + \frac{\mathrm{v}^2k_0^2}{\mathrm{v}^2}\right)\psi = 0$$

And therefore the above form does satisfy the wave equation for large ρ*.*

5. Our simple model of dispersion assumed that the response of an electron to a disturbance from its equilibrium position is a decaying oscillation, which we called the impulse response of the electron, e.g.,

$$h\left[t\right] = A_0 \exp\left[-\gamma_0 t\right] \cdot STEP\left[t\right] \cdot \sin\left[2\pi\nu_0 t\right]$$

From this we derived the electron amplitude as a function of the frequency ν of the incident light; we called it $H\left[\nu\right]$, and we outlined how this leads to the index of refraction. Here, assume that the impulse response of the electron motion to a disturbance includes two different sinusoidal frequencies that decay from different amplitudes at different rates. Evaluate and plot the frequency response of this system as real and imaginary parts, and as magnitude and phase.



(a), (b) impulse responses of two decaying exponentials where the natural oscillation frequency of (b) is much larger than that of (a); (c) sum of the two impulse responses; (d) real and imaginary parts of the Fourier transform of (c), showing that the two oscillations are at $\frac{1}{2}$ and 4 cycles per unit time; (e) magnitude; (f) phase, showing that the driven oscillation is approximately in phase for $\nu < 4$ and out of phase for $\nu > 4$ cycles per unit time.