

SIMG-712-01-20042 Solution Set #1

1. A source of harmonic motion of the form $y(t) = 6 \cdot \cos(\omega_0 t)$ located at the origin of the spatial coordinate system emits a wave that travels through a uniform (i.e., homogeneous) medium at a rate of 60 mm per second.

(HINT: It is ALWAYS useful to draw a diagram of the problem before trying to solve it!)

- a. Find the formula for the displacement due to this wave at a distance of 800 mm from the origin.

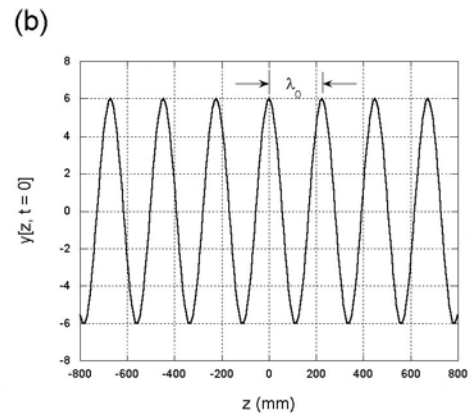
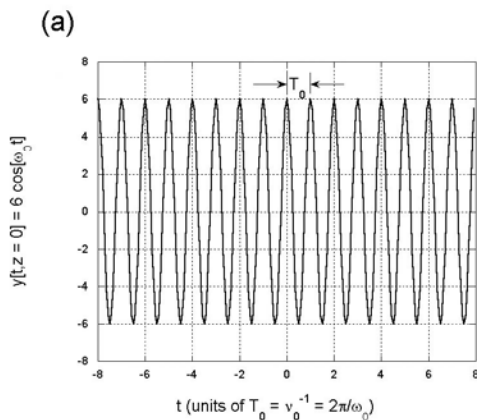
This is a “travelling wave” which has a sinusoidal form in both time AND space. We know that $v = \frac{\omega_0}{k_0} = 60 \frac{\text{mm}}{\text{s}}$. The temporal part of the wave at the origin must have the same functional form as $y[t]$:

$$y[z = 0, t] = 6 \cdot \cos(\omega_0 t) = 6 \cdot \cos(2\pi\nu_0 t)$$

where $\nu_0 = \frac{\omega_0}{2\pi} = \frac{v}{\lambda_0}$. From the statement of the problem, the amplitude of the function that was located at the origin at $t = 0$ will travel to the point located 60 mm away at $t = 1$ s. The amplitude will travel to the point located 800 mm away in $t = \frac{800 \text{ mm}}{60 \frac{\text{mm}}{\text{s}}} = \frac{40}{3}$ s. Consult the sketch (a) to see the amplitude evaluated as a function of time at the origin ($z = 0$). Sketch (b) shows the amplitude as a function of position z at $t = 0$ s, i.e., it is a “snapshot” of the spatial wave at this one time. Note that this wave has a “spatial period” labeled by λ_0 , the wavelength of the spatial oscillation. The amplitude (displacement) of the wave at the distance $z = 800$ mm is:

$$y[z = 800 \text{ mm}, t] = 6 \cdot \cos\left(2\pi\nu_0 t - 2\pi \frac{800 \text{ mm}}{\lambda_0 [\text{mm}]}\right)$$

where λ_0 is not specified.



We do know the relationship for the velocity, wavelength, and temporal frequency:

$$v_0 = \lambda_0 \cdot \nu_0 \Rightarrow \lambda_0 = \frac{v_0}{\nu_0}$$

$$\Rightarrow y[z, t] = A_0 \cdot \cos\left(2\pi\nu_0 t - 2\pi\nu_0 \frac{z}{v_0}\right)$$

$$y[z, t] = 6 \cdot \cos\left(\omega_0\left(t - \frac{800 \text{ mm}}{v_0 \lambda_0 \left[\frac{\text{mm}}{\text{s}}\right]}\right)\right)$$

$$= 6 \cdot \cos\left(\omega_0\left(t - \frac{800 \text{ mm}}{v_0 \left[\frac{\text{mm}}{\text{s}}\right]}\right)\right)$$

$$= 6 \cdot \cos\left(\omega_0\left(t - \frac{800 \text{ mm}}{60 \frac{\text{mm}}{\text{s}}}\right)\right)$$

$$y[z = 800 \text{ mm}, t] = 6 \cdot \cos\left(\omega_0\left(\frac{800 \text{ mm}}{60 \frac{\text{mm}}{\text{s}}} - t\right)\right)$$

$$= 6 \cdot \cos\left(\omega_0\left(\frac{800}{60} \text{ s} - t\right)\right)$$

- b.** Find the displacement at that distance for $t = 60 \text{ s}$.

Just substitute this time into the equation in part (a):

$$y[z = 800 \text{ mm}, t = 60 \text{ s}] = 6 \cdot \cos\left(\omega_0\left(\frac{800 \text{ mm}}{60 \frac{\text{mm}}{\text{s}}} - 60 \text{ s}\right)\right)$$

- 2.** What is the phase difference (in radians) between any two points on a harmonic electromagnetic wave separated by $\Delta z = 1 \mu\text{m}$ (1 micron) if the wavelength is $\lambda = 550 \text{ nm}$?

Though the statement did not specify, we can assume that the wave is in vacuum.

$$\Delta\phi[\text{radians}] = 2\pi \frac{\Delta z}{\lambda} = 2\pi \frac{1 \mu\text{m}}{550 \text{ nm}} \cong 11.42 \text{ radians}$$

3. N simple temporal harmonic oscillatory motions ($N > 1$) with the same amplitude and temporal frequency are superimposed (summed), i.e., the output $g[t]$ may be written as:

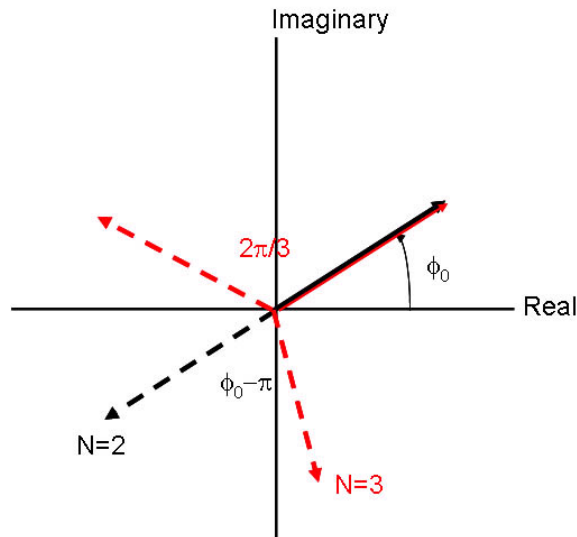
$$g[t] = \sum_{n=1}^N A_0 \cos[2\pi\nu_0 t + \phi_n]$$

The phase difference between successive pairs of oscillations is:

$$\Delta\phi_n = \phi_n - \phi_{n-1}, \text{ where } n = 2, 3, \dots, N$$

If the phase difference between each successive pair is invariant, find one value of this phase difference as a function of N for which the amplitude of the sum is zero. Again, it may be useful to draw a picture to help you solve the problem, and it also may be useful to consider the result for small integer values of N .

This is easy to see on the Argand (“phasor”) diagram. The initial phase for the first oscillator is ϕ_0 . If $N = 2$, then the second oscillator must have the same amplitude but be “out of phase” relative to the first, i.e., $\phi_1 = \phi_0 \pm \pi$, as shown on the sketch. If $N = 3$, then the sum of the second and third oscillations must create a term with the same amplitude and “out of phase.” In general, the initial phase of the second oscillation could have the angle $\phi_0 + \frac{2\pi}{N}$, where N is the number of oscillators to be added. So the phase of the n^{th} oscillator is $\phi_n = \phi_0 + \frac{2\pi}{N} \cdot n$.



4. Use the Euler relation: $e^{i\theta} = \cos[\theta] + i \sin[\theta]$ to derive expressions for the following in terms of $\cos[\theta_1]$, $\sin[\theta_1]$, $\cos[\theta_2]$, and $\sin[\theta_2]$:

a. $\sin[2\theta_1]$

$$\sin[2\theta_1] = \text{Im}\{\exp[+i(2\theta_1)]\}$$

we know that $\exp[+i\theta_1] = \cos[\theta_1] + i \sin[\theta_1]$

$$(\exp[+i\theta_1])^2 = \exp[+i2\theta_1]$$

$$(\exp[+i\theta_1])^2 = (\cos[\theta_1] + i \sin[\theta_1])^2$$

$$\exp[+i(2\theta_1)] = \cos^2[\theta_1] + i^2 \sin^2[\theta_1] + 2i \cos[\theta_1] \sin[\theta_1]$$

$$= (\cos^2[\theta_1] - \sin^2[\theta_1]) + i(2 \cos[\theta_1] \sin[\theta_1])$$

$$\text{Im}\{\exp[+i(2\theta_1)]\} = \sin[2\theta_1] = 2 \cos[\theta_1] \sin[\theta_1]$$

also: $\text{Re}\{\exp[+i(2\theta_1)]\} = \cos[2\theta_1] = \cos^2[\theta_1] - \sin^2[\theta_1]$

b. $\cos[\theta_1 \pm \theta_2]$

$$\cos[\theta_1 \pm \theta_2] = \text{Re}\{\exp[i(\theta_1 \pm \theta_2)]\}$$

$$\exp[i(\theta_1 \pm \theta_2)] = \exp[i\theta_1] \cdot \exp[\pm i\theta_2]$$

$$= (\cos[\theta_1] + i \sin[\theta_1]) \cdot (\cos[\theta_2] \pm i \sin[\theta_2])$$

$$= \cos[\theta_1] \cos[\theta_2] + i \cos[\theta_1] \sin[\theta_2] \pm i \sin[\theta_1] \cos[\theta_2] \mp \sin[\theta_1] \sin[\theta_2]$$

$$= (\cos[\theta_1] \cos[\theta_2] \mp \sin[\theta_1] \sin[\theta_2]) + i(\cos[\theta_1] \sin[\theta_2] \pm \sin[\theta_1] \cos[\theta_2])$$

$$\text{Re}\{\exp[i(\theta_1 \pm \theta_2)]\} = \cos[\theta_1 \pm \theta_2] = \cos[\theta_1] \cos[\theta_2] \mp \sin[\theta_1] \sin[\theta_2]$$

also: $\text{Im}\{\exp[i(\theta_1 \pm \theta_2)]\} = \sin[\theta_1 \pm \theta_2] = \cos[\theta_1] \sin[\theta_2] \pm \sin[\theta_1] \cos[\theta_2]$

c. $\sin[\theta_1 \pm \theta_2]$

Obtained in part (b)

5. Use the formulas for $\cos[\theta_1 \pm \theta_2]$ and for $\sin[\theta_1 \pm \theta_2]$ just derived to derive expressions for the following in terms of the sum and difference frequencies $\omega_1 \pm \omega_2$. ALSO plot the results for $\omega_1 = 1$ and $\omega_2 = \frac{3}{2}$ radian per second, respectively.

From the previous problem, we have that:

$$\cos[\theta_1 \pm \theta_2] = \cos[\theta_1]\cos[\theta_2] \mp \sin[\theta_1]\sin[\theta_2]$$

$$\sin[\theta_1 \pm \theta_2] = \cos[\theta_1]\sin[\theta_2] \pm \sin[\theta_1]\cos[\theta_2]$$

The second terms in the cosine expressions have the desired forms:

$$\cos[\theta_1 - \theta_2] = \cos[\theta_1]\cos[\theta_2] + \sin[\theta_1]\sin[\theta_2]$$

$$\cos[\theta_1 + \theta_2] = \cos[\theta_1]\cos[\theta_2] - \sin[\theta_1]\sin[\theta_2]$$

Subtract these two:

$$\cos[\theta_1 - \theta_2] - \cos[\theta_1 + \theta_2] = 2\sin[\theta_1]\sin[\theta_2]$$

$$\Rightarrow \sin[\theta_1]\sin[\theta_2] = \frac{1}{2}\cos[\theta_1 - \theta_2] - \frac{1}{2}\cos[\theta_1 + \theta_2]$$

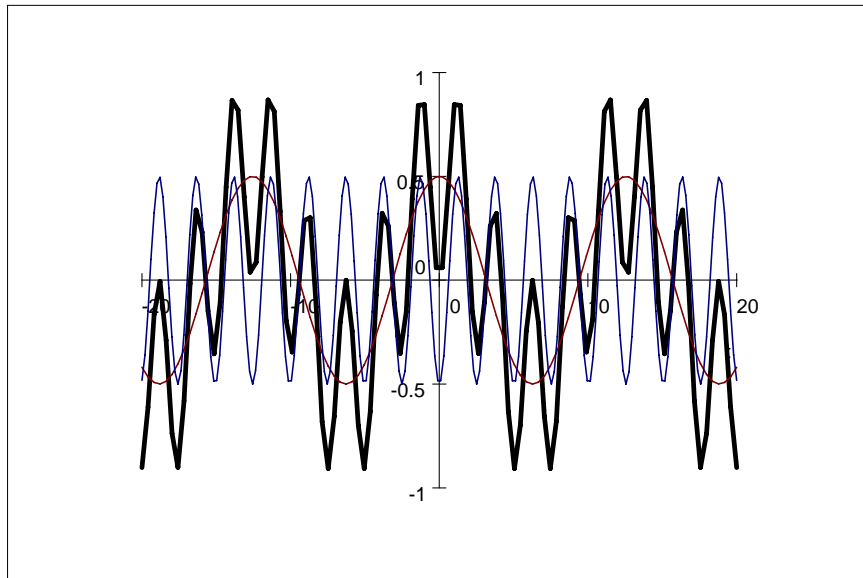
If these two are added, we get the expression for $\cos[\theta_1]\cos[\theta_2]$:

$$\cos[\theta_1]\cos[\theta_2] = \frac{1}{2}\cos[\theta_1 - \theta_2] + \frac{1}{2}\cos[\theta_1 + \theta_2]$$

a. $\sin[\omega_1 t] \cdot \sin[\omega_2 t]$

$$\begin{aligned} \sin[\omega_1 t] \cdot \sin[\omega_2 t] &= \frac{1}{2}\cos[\omega_1 t - \omega_2 t] - \frac{1}{2}\cos[\omega_1 t + \omega_2 t] \\ &= \frac{1}{2}\cos[(\omega_1 - \omega_2)t] - \frac{1}{2}\cos[(\omega_1 + \omega_2)t] \end{aligned}$$

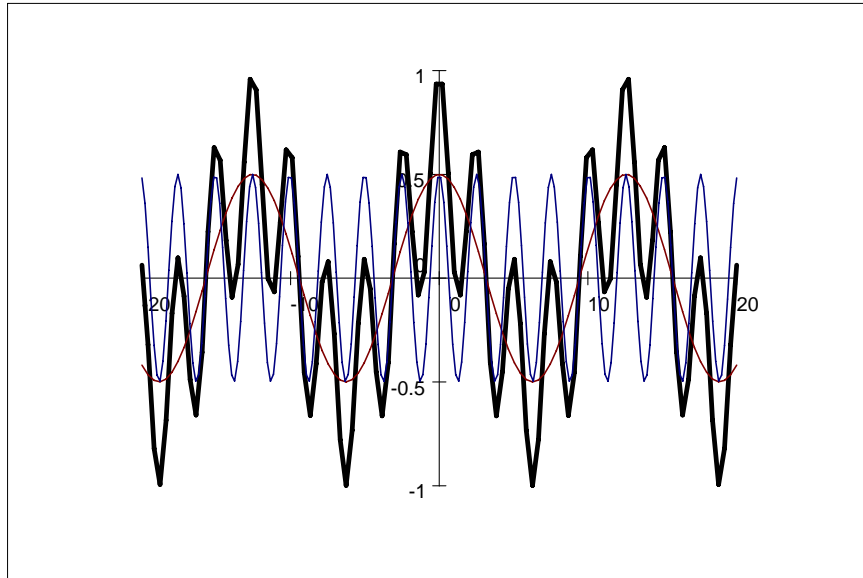
$$\sin[t] \cdot \sin\left[\frac{3}{2}t\right] = \frac{1}{2}\cos\left[-\frac{1}{2}t\right] - \frac{1}{2}\cos\left[\frac{5}{2}t\right] = \frac{1}{2}\cos\left[\frac{t}{2}\right] - \frac{1}{2}\cos\left[\frac{5}{2}t\right]$$



$\sin[t] \cdot \sin\left[\frac{3}{2}t\right]$, (note that the amplitude at $t=0$ is a minimum).

b. $\cos[\omega_1 t] \cdot \cos[\omega_2 t]$

$$\begin{aligned}\cos[\omega_1 t] \cdot \cos[\omega_2 t] &= \frac{1}{2} \cos[\omega_1 t - \omega_2 t] + \frac{1}{2} \cos[\omega_1 t + \omega_2 t] \\ &= \frac{1}{2} \cos[(\omega_1 - \omega_2)t] + \frac{1}{2} \cos[(\omega_1 + \omega_2)t]\end{aligned}$$



$\cos[t] \cdot \cos[\frac{3}{2}t] = \frac{1}{2} \cos[\frac{t}{2}] + \frac{1}{2} \cos[\frac{5t}{2}]$, plotted in black; $\frac{1}{2} \cos$

c. $\sin[\omega_1 t] \pm \sin[\omega_2 t]$

From previous problem:

$$\sin[(\omega_1 \pm \omega_2)t] = \cos[\omega_1 t] \sin[\omega_2 t] \pm \sin[\omega_1 t] \cos[\omega_2 t]$$

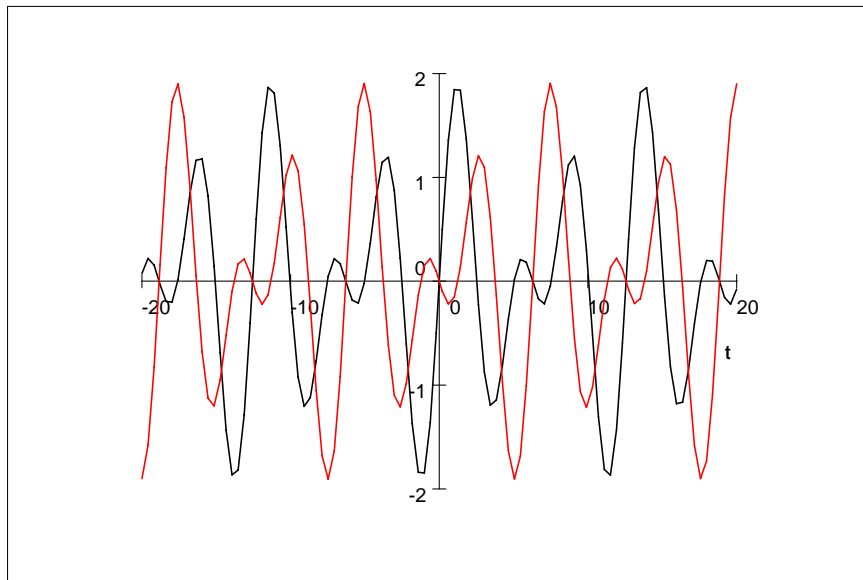
$$\begin{aligned} \sin[\omega_1 t] &= \sin\left[\frac{(\omega_1 + \omega_2)t + (\omega_1 - \omega_2)t}{2}\right] \\ &= \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right] + \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right] \end{aligned}$$

$$\begin{aligned} \sin[\omega_2 t] &= \sin\left[\frac{(\omega_1 + \omega_2)t - (\omega_1 - \omega_2)t}{2}\right] \\ &= \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right] - \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right] \end{aligned}$$

$$\sin[\omega_1 t] + \sin[\omega_2 t] = 2 \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right]$$

$$\sin[\omega_1 t] - \sin[\omega_2 t] = 2 \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right]$$

$$\Rightarrow \sin[\omega_1 t] \pm \sin[\omega_2 t] = 2 \sin\left[\left(\frac{\omega_1 \pm \omega_2}{2}\right)t\right] \cos\left[\left(\frac{\omega_1 \mp \omega_2}{2}\right)t\right]$$



$\sin[t] \pm \sin\left[\frac{3}{2}t\right]$ (“+” in solid black, “-” in red)

d. $\cos[\omega_1 t] \pm \cos[\omega_2 t]$

$$\cos[\omega_1 t] = \cos\left[\frac{(\omega_1 + \omega_2)t + (\omega_1 - \omega_2)t}{2}\right]$$

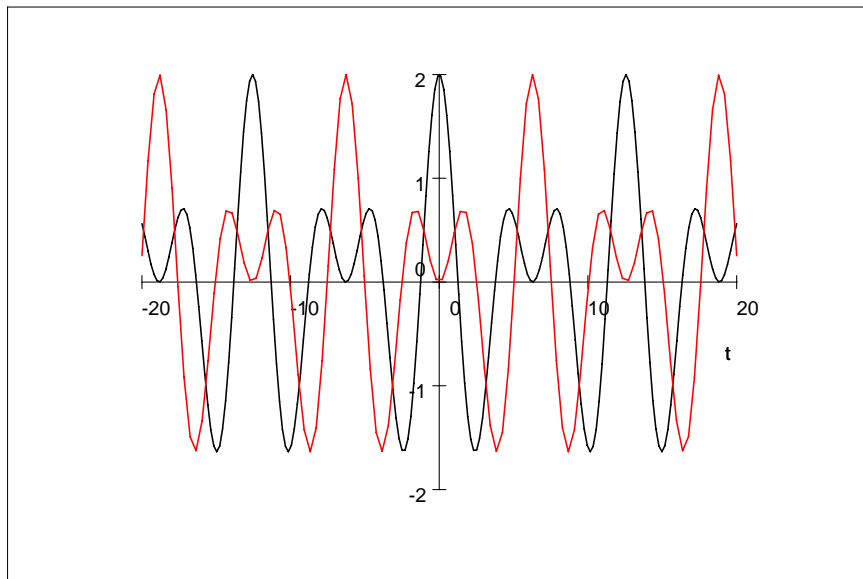
$$= \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right] - \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right]$$

$$\cos[\omega_2 t] = \cos\left[\frac{(\omega_1 + \omega_2)t - (\omega_1 - \omega_2)t}{2}\right]$$

$$= \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right] + \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right]$$

$$\cos[\omega_1 t] + \cos[\omega_2 t] = 2 \cos\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \cos\left[\frac{(\omega_1 - \omega_2)t}{2}\right]$$

$$\cos[\omega_1 t] - \cos[\omega_2 t] = -2 \sin\left[\frac{(\omega_1 + \omega_2)t}{2}\right] \sin\left[\frac{(\omega_1 - \omega_2)t}{2}\right]$$



$\cos[t] \pm \cos\left[\frac{3}{2}t\right]$ (“+” in solid black, “-” in dashed red)

6. Consider the superposition of two sinusoidal traveling waves:

$$f_1[z, t] = A_1 \cos[k_1 z - \omega_1 t],$$

$$A_1 = 10 \text{ mm}, \quad v_1 = 1000 \text{ Hz}, \quad v_1 = 250 \frac{\text{m}}{\text{s}}$$

$$f_2[z, t] = A_2 \cos[k_2 z - \omega_2 t],$$

$$A_2 = 9 \text{ mm}, \quad v_2 = 1500 \text{ Hz}, \quad v_2 = 500 \frac{\text{m}}{\text{s}}$$

- Find an expression for the resulting wave in terms of the average wave, the modulation wave, plus any remaining amplitude.
- Calculate the wavelengths of the average and modulation waves.
- Find the velocities of the average and modulation waves.
- Does this system exhibit normal or anomalous dispersion?

$$\lambda_1 = \frac{v_1}{v_1} = \frac{250 \frac{\text{m}}{\text{s}}}{1000 \text{ Hz}} = \frac{1}{4} \text{ m} \Rightarrow k_1 = \frac{2\pi}{\lambda_1} = 8\pi \text{ m}^{-1}$$

$$\omega_1 = 2\pi v_1 = 2\pi \cdot 1000 \text{ Hz} = 2000\pi \text{ s}^{-1}$$

$$\lambda_2 = \frac{v_2}{v_2} = \frac{500 \frac{\text{m}}{\text{s}}}{1500 \text{ Hz}} = \frac{1}{3} \text{ m} \Rightarrow k_2 = 6\pi \text{ m}^{-1}$$

$$\omega_2 = 2\pi v_2 = 2\pi \cdot 1500 \text{ Hz} = 3000\pi \text{ s}^{-1}$$

So right away we see that the higher-frequency wave travels faster, so we expect that the medium exhibits anomalous dispersion.

$$\begin{aligned} & (10 \text{ mm}) \cos[k_1 z - \omega_1 t] + (9 \text{ mm}) \cos[k_2 z - \omega_2 t] \\ &= (1 \text{ mm}) \cos[k_1 z - \omega_1 t] + (9 \text{ mm}) (\cos[k_1 z - \omega_1 t] + \cos[k_2 z - \omega_2 t]) \\ &= (1 \text{ mm}) \cos[k_1 z - \omega_1 t] + 2 \cdot (9 \text{ mm}) (\cos[k_{avg} z - \omega_{avg} t] \cdot \cos[k_{mod} z - \omega_{mod} t]) \end{aligned}$$

This is the sum of a traveling wave with velocity v_1 and average and modulation waves. Consider these latter two individually:

$$\cos[k_{avg} z - \omega_{avg} t] = \cos\left[\left(\frac{k_1 + k_2}{2}\right)z - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right]$$

$$k_{avg} = \frac{1}{2}(8\pi \text{ m}^{-1} + 6\pi \text{ m}^{-1}) = 7\pi \text{ m}^{-1}$$

$$\Rightarrow \lambda_{avg} = \frac{2\pi}{k_{avg}} = \frac{2}{7} \text{ m} \cong 0.286 \text{ m}$$

$$\omega_{avg} = \frac{\omega_1 + \omega_2}{2} = \frac{2000\pi \text{ s}^{-1} + 3000\pi \text{ s}^{-1}}{2} = 2500\pi \text{ s}^{-1}$$

$$v_{avg} = \frac{\omega_{avg}}{k_{avg}} = \frac{2500\pi \text{ s}^{-1}}{7\pi \text{ m}^{-1}} = \frac{2500}{7} \frac{\text{m}}{\text{s}} \cong 357.1 \frac{\text{m}}{\text{s}}$$

$$\begin{aligned} \cos[k_{\text{mod}}z - \omega_{\text{mod}}t] &= \cos\left[\left(\frac{k_1 - k_2}{2}\right)z - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \\ k_{\text{mod}} &= \frac{1}{2}(8\pi \text{ m}^{-1} - 6\pi \text{ m}^{-1}) = \pi \text{ m}^{-1} \\ \Rightarrow \lambda_{\text{mod}} &= \frac{2\pi}{k_{\text{mod}}} = 2 \text{ m} \\ \omega_{\text{mod}} &= \frac{\omega_1 - \omega_2}{2} = \frac{2000\pi \text{ s}^{-1} - 3000\pi \text{ s}^{-1}}{2} = -1000\pi \text{ s}^{-1} \\ v_{\text{mod}} &= \frac{\omega_{\text{mod}}}{k_{\text{mod}}} = \frac{-1000\pi \text{ s}^{-1}}{\pi \text{ m}^{-1}} = -1000 \frac{\text{m}}{\text{s}} \end{aligned}$$

The magnitude of the modulation velocity is larger than the magnitude of the average velocity, but the modulation velocity is negative (the modulation wave moves to the left, whereas the average wave moves to the right!) What does this mean? I'd say that it shows that this is an unphysical situation!

7. The phase velocity of waves in some medium is proportional to $\omega^{+\frac{1}{4}}$. Find an expression for the modulation velocity and determine whether the waves exhibit normal or anomalous dispersion.

$$\begin{aligned} v_{\phi} &= \frac{\omega}{k} \propto \omega^{+\frac{1}{4}} \Rightarrow k \propto \omega^{+\frac{3}{4}} \Rightarrow \omega \propto k^{+\frac{4}{3}} \\ v_{\text{mod}} &= \frac{d\omega}{dk} \propto \frac{d}{dk} \left(k^{+\frac{4}{3}}\right) = +\frac{4}{3}k^{+\frac{1}{3}} \\ v_{\phi} &= \frac{\omega}{k} \propto \frac{k^{+\frac{4}{3}}}{k} = k^{+\frac{1}{3}} \\ v_{\text{mod}} &\propto +\frac{4}{3}k^{+\frac{1}{3}} > v_{\phi} \propto k^{+\frac{1}{3}} \Rightarrow v_{\text{mod}} > v_{\phi} \\ &\Rightarrow \text{anomalous dispersion} \end{aligned}$$