SIMG-712 Solutions to Midterm Exam

1. A lens system is composed of two thin lenses separated by a variable distance t that will be used in air. The prescriptions for the surfaces of the two lenses are:

$$\begin{aligned} L_1 &: n = 1.5 \\ R_1 &= +500\,\text{mm} \\ R_2 &= +200\,\text{mm} \end{aligned}$$

$$\begin{aligned} L_2 &: n = 1.6 \\ R_1 &= -100\,\text{mm} \\ R_2 &= -200\,\text{mm} \end{aligned}$$

   (a) Find the focal lengths of the two lenses.

lensmaker's equation for lens in air:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$f_1 = \left((1.5-1)\left(\frac{1}{+500\,\text{mm}} - \frac{1}{200\,\text{mm}}\right)\right)^{-1} = \left(0.5 \cdot -\frac{3}{1000\,\text{mm}}\right)^{-1} = \frac{1}{-0.0015}\,\text{mm}$$

$$\boxed{f_1 = -666\tfrac{2}{3}\,\text{mm} = -\tfrac{2}{3}\,\text{m}}$$

$$f_2 = \left((1.6-1)\left(\frac{1}{-100\,\text{mm}} - \frac{1}{-200\,\text{mm}}\right)\right)^{-1} = \left(0.6 \cdot -\frac{1}{200\,\text{mm}}\right)^{-1}$$

$$\boxed{f_2 = -333\tfrac{1}{3}\,\text{mm} = -\tfrac{1}{3}\,\text{m}}$$

   (b) Find the focal length of these two lenses if in contact.

power of a thin-lens system:

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t \implies f = \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}\right)^{-1}$$

$$t = 0 \implies f = \left(\frac{1}{-666\frac{2}{3}\,\text{mm}} + \frac{1}{-333\frac{1}{3}\,\text{mm}}\right)^{-1} = \boxed{f_{eff} = -\tfrac{2000}{9}\,\text{mm} \cong -222.22\,\text{mm}}$$

(c) Characterize the image of an object that is 20 mm tall and 2 mm "deep" (dimension along the direction of the optical axis. The object is located such that $\overline{\mathbf{OV}} = 250\,\text{mm}$.

$$
\begin{aligned}
s' &= \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{-\frac{2000}{9}\,\text{mm}} - \frac{1}{250\,\text{mm}}\right)^{-1} \\
& \boxed{s' = -\tfrac{2000}{17}\,\text{mm} \cong -117.65\,\text{mm}} \\
M_T &= -\frac{-\frac{2000}{17}\,\text{mm}}{250\,\text{mm}} = +\frac{8}{17} \implies \textit{image is upright} \\
\textit{height of image is} &: \\
20\,\text{mm} \cdot M_T &= 20\,\text{mm} \cdot \frac{8}{17} = \frac{160}{17}\,\text{mm} \cong 9.41\,\text{mm} \\
M_L &= -\left(M_T\right)^2 = -\left(\frac{8}{17}\right)^2 = -\frac{64}{289} \cong -0.22 \\
\textit{``depth'' of image is approximately} &: \\
2\,\text{mm} \cdot M_L &= 2\,\text{mm} \cdot -\frac{64}{289} = -\frac{128}{289}\,\text{mm} \cong -0.44\,\text{mm}
\end{aligned}
$$

(d) Find the separation t such that the power of the lens system is 0 diopters. Sketch the system including a ray entering parallel to the optical axis.

$$\varphi = \varphi_1 + \varphi_2 - \varphi_1\varphi_2 t$$

$$\varphi = 0 \implies f = \infty \implies t = \frac{\varphi_1 + \varphi_2}{\varphi_1\varphi_2}$$

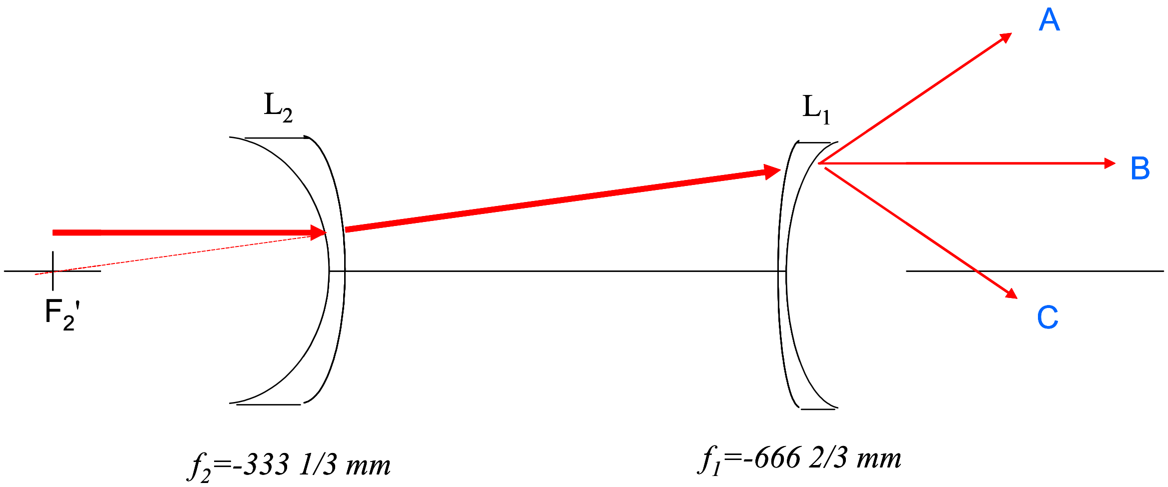
$$t = \frac{\left(-3\,\text{m}^{-1}\right) + \left(-\frac{3}{2}\,\text{m}^{-1}\right)}{\left(-3\,\text{m}^{-1}\right)\left(-\frac{3}{2}\,\text{m}^{-1}\right)} = \frac{-\frac{9}{2}\,\text{m}^{-1}}{-\frac{9}{2}\,\text{m}^{-2}} = -1\,\text{m}$$

$$\boxed{t = -1000\,\text{mm} \quad (?!!??)}$$

BUT: a separation with a negative distance has no meaning! This would imply that the first element is behind the second element! In short:

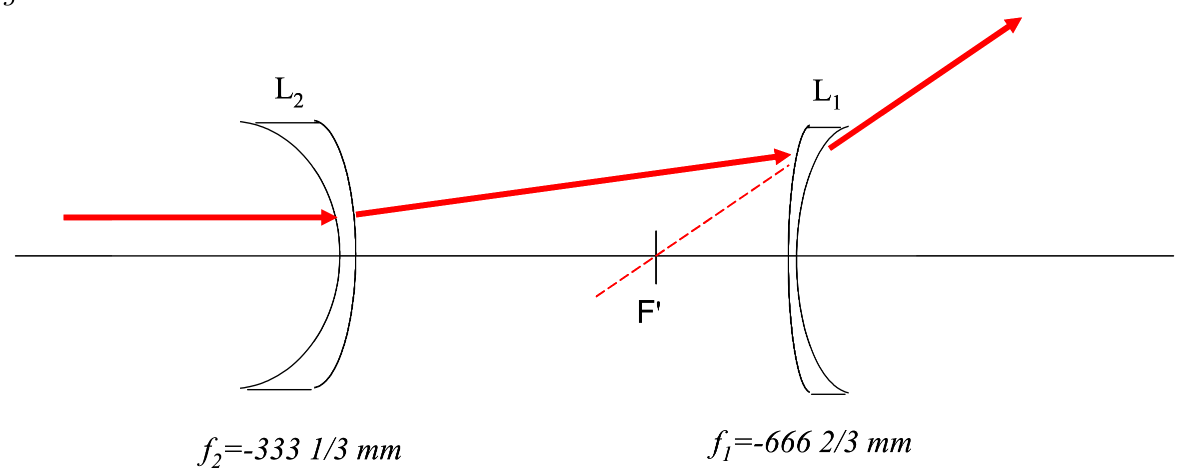
| *You Can't Make a Telescope from Two Negative Lenses!* |

(no need to make a drawing, but I'll do it anyway for illustration. If you assumed that $t = -1000\,\text{mm}$ *means that lens 2 is in front of lens 1 (as many of you did), do the sketch:*



Where does the ray out of the second lens ("L_1" since the order of the lenses was reversed)? Direction **B** *is where you want it to go if the system is to have infinite focal length, but this would require the lens to make the rays* ***converge*** *rather than* ***diverge****, as a negative lens would do. To make the ray go in the direction* ***C*** *would require that the focal length of L_1 be positive but even shorter (more powerful lens) than to get ray* ***B****. The ray must diverge from a negative, thus the only choice is direction* ***A****.*

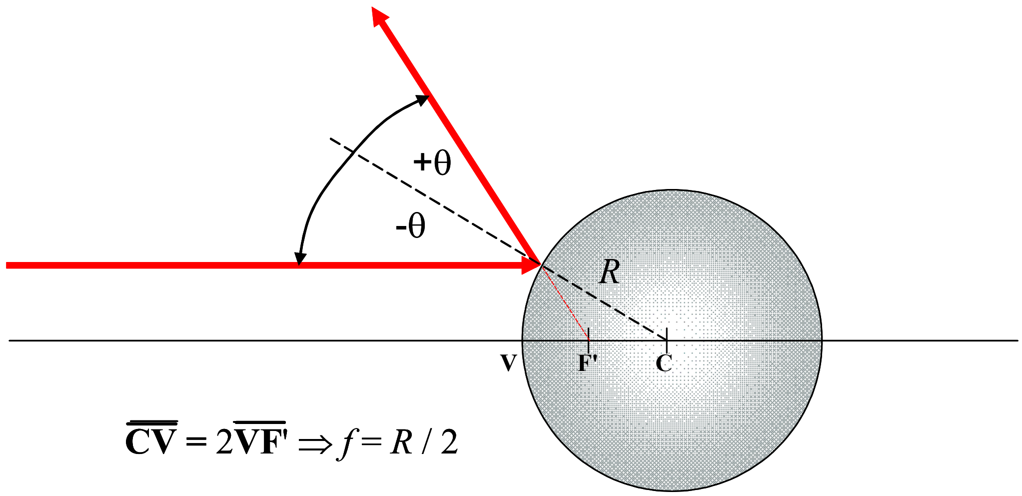
Had the lenses been placed in the original order, the same situation would result. The image-space focal point of a system created from two negative lenses is virtual, the power is negative. This demonstrates that you can't make a telescope from two negative lenses.



2. A reflective sphere (imagine a ball bearing) with diameter 50 mm acts as a spherical mirror that can be used to image objects:

   (a) Determine the focal length of the "system."

   Draw the picture and use the reflection law ($\theta \to -\theta$)



$$d = 50\,\text{mm} \implies R = 25\,\text{mm}$$
$$\implies \textit{paraxial focal length } \boxed{f = -\frac{R}{2} = -12.5\,\text{mm}}$$

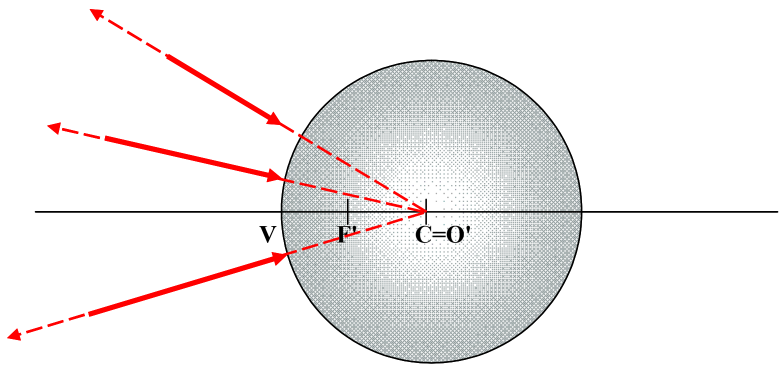
   which can be inferred from the sketch (this is a recurring theme – $\boxed{\textit{MAKE A SKETCH!}}$

   (b) Sketch the "system," including the location of the image-space focal and principal points.

   The focal point is shown above. What about the image-space principal point? It's the point that is one focal length away from the image-space focal point. From the drawing you can see that **H′** *coincides with* **V.** *This makes sense because it is the location of the image point if the transverse magnification is* $M_T = +1$*. By analogy with the thin lens (where the principal points coincide with* **V** *and* **V′**, *the principal points of the mirror coincide with* **V′** *(and thus with* **V**).

   (c) Determine the location of the input object that produces a paraxial image at the center of the sphere.

   Again, Draw the Picture!!! Since it is a mirror system, the image will be virtual. The incoming rays must be "aimed" at the center of the sphere to appear to emerge from the center of the sphere.



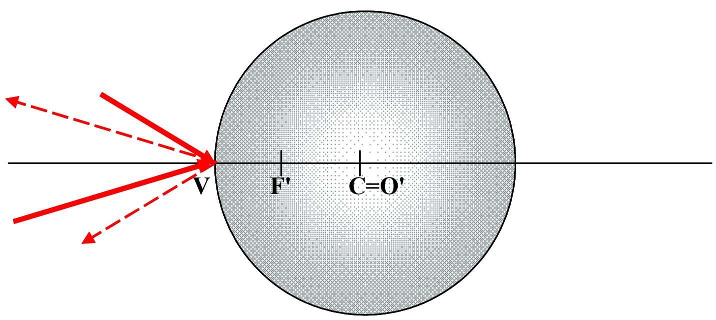
The distance can also be evaluated from the Gaussian imaging equation with $\overline{\mathbf{OV}} = -R = -25\,\text{mm}$:

$$s = \left(\frac{1}{f} - \frac{1}{s'}\right)^{-1} = \left(\frac{1}{-12.5\,\text{mm}} - \frac{1}{-25\,\text{mm}}\right)^{-1} = \boxed{-25.0\,\text{mm} = s}$$

which also is at the center of the sphere!

(d) Determine the location of the input object such that the paraxial image is located at the vertex of the mirror.

Here the paraxial object is located at the objects-space principal point $\mathbf{H}$*, which coincides with* $\mathbf{V}, \mathbf{V}'$*, and* $\mathbf{H}'$*. Again, the relationship between the angles of incidence and reflection for the paraxial ray shows that the rays appear to emerge from the same location.*



You have a bit of a problem evaluating this image distance from the imaging equation because $s = 0$*, but you can solve it if you just plow through it:*

$$s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{-12.5\,\text{mm}} - \frac{1}{0\,\text{mm}}\right)^{-1} = (-\infty)^{-1} = \boxed{0 = s'}$$

(e) Determine the transverse magnification for the object-image combination in part (d).

The transverse magnification "appears to be"

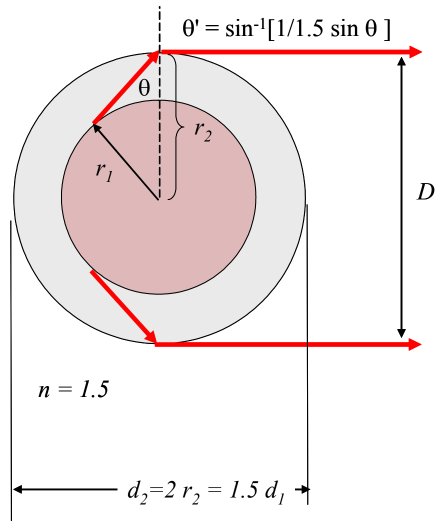
$$M_T = -\frac{s'}{s} = -\frac{0}{0} = ?$$

but what this really means is that the object and image are located at the object-space and image-space principal points,which coincide with each other and with the vertices,Therefore $\boxed{M_T = +1}$

(f) Sketch the object, system, and image in the configuration in part (d).

done in part (d)

3. A mercury thermometer is constructed from a cylindrical glass tube ($n = 1.5$). The outer diameter is $\frac{3}{2}$ larger than the inner diameter of the tube. The outer diameter is *much* smaller than the viewing distance, which means that the rays reaching the eye are approximately parallel. Determine the apparent diameter of the mercury column (i.e., the inner tube diameter) relative to the apparent outside diameter. HINT: sketch the entire "system" first.



$$n = 1.5 \implies 1.5 \cdot \sin[\theta] = 1 \cdot \sin[\theta'] \implies \sin[\theta'] = 1.5 \cdot \sin[\theta]$$

From sketch:

$$\sin[\theta] = \frac{r_1}{r_2} = \frac{r_1}{1.5 \cdot r_1} = \frac{1}{1.5}$$

$$\implies \sin[\theta'] = 1.5 \cdot \frac{1}{1.5} = 1 \implies \theta' = \sin^{-1}[1] = \frac{\pi}{2}$$

$$D = 2 \cdot r_2 \sin[\theta'] = 2r_2$$

Because the "half angle" $\theta' = \frac{\pi}{2}$, *the "full angle" is* π,

so the image of the column fills the outer tube

The observed diameter D of the column is equal to the diameter of the larger tube.

4. The magnitude of the electric field for spherical electromagnetic waves emanating from a point source at the origin may be written

$$E[r,t] = \frac{E_0}{r}\cos[k_0 r - \omega_0 t]$$

where E_0 is a constant and r is the radial coordinate. With a short calculation based on the definition of the Poynting vector, show that E MUST vary as r^{-1}.

We know that energy must be conserved, so the flux of energy per unit time flowing across any spherical surface centered at the source must be the same.

$$\frac{\textit{energy}}{\textit{time}} = \textit{constant}$$

The energy flux per unit area per unit time is the time average of the Poynting vector $\underline{\mathbf{s}}$*:*

$$\frac{\textit{energy}}{\textit{unit area, unit time}} = \langle|\underline{\mathbf{s}}|\rangle \Longrightarrow \iint_{\text{spherical surface}} \langle|\underline{\mathbf{s}}|\rangle \bullet d\underline{\mathbf{A}} = b \text{ (a constant)}$$

The vector $\underline{\mathbf{s}}$ *is constant across the surface and parallel to* $\underline{\mathbf{A}}$*, so its time average may be extracted from the integral:*

$$\begin{aligned}\iint_{\text{spherical surface}} \langle|\underline{\mathbf{s}}|\rangle \bullet d\underline{\mathbf{A}} &= \langle|\underline{\mathbf{s}}|\rangle \iint_{\text{spherical surface}} d\underline{\mathbf{A}} \\ &= \langle|\underline{\mathbf{s}}|\rangle \cdot 4\pi r^2 = b\end{aligned}$$

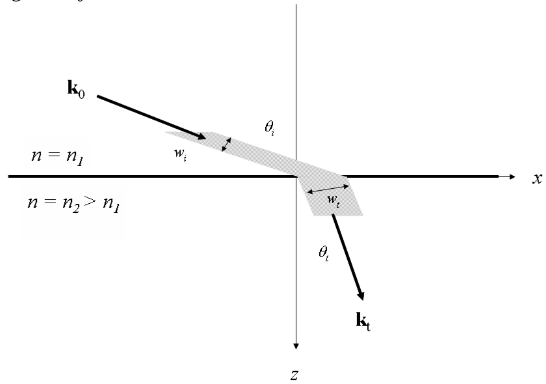
We also know that $|\underline{\mathbf{s}}| \propto |\underline{\mathbf{E}}_0|\,|\underline{\mathbf{B}}_0| \propto |\underline{\mathbf{E}}_0|^2 = E_0^2$*:*

$$E_0^2 \cdot 4\pi r^2 = b \Longrightarrow |\underline{\mathbf{E}}[r,t]|^2 \propto \frac{|\underline{\mathbf{E}}_0|^2}{r^2} \Longrightarrow |\underline{\mathbf{E}}[r,t]| \propto \frac{|\underline{\mathbf{E}}_0|}{r} = \frac{E_0}{r}$$

5. Regardless of the polarization, the *reflectance* of a material at an interface is just the square of the amplitude reflectance coefficient, but the *transmission* at the interface includes an additional multiplicative factor than just the square of the amplitude transmittance coefficient.

(a) Explain why this is so and illustrate your answer with sketches.

The refraction at the interface changes the "width" of the beam in one direction, so that the area of the transmitted beam is different from that of the incident beam. This is illustrated in the figure for a case with $n_1 > n_2$:



(b) Derive the additional multiplicative factor.

The area of the transmitted beam changes in proportion to the dimension along the x-axis in this case, which allows us to see that:

$$\frac{A_2}{A_1} = \frac{w_2}{w_1} = \frac{\sin\left[\frac{\pi}{2} - \theta_t\right]}{\sin\left[\frac{\pi}{2} - \theta_0\right]} = \frac{\cos\left[\theta_t\right]}{\cos\left[\theta_0\right]}$$

which leads to the final expression for the transmission at the interface:

$$T = \frac{n_2}{n_1} \cdot t^2 \cdot \left(\frac{\cos\left[\theta_t\right]}{\cos\left[\theta_0\right]}\right)$$

$$\boxed{T = \left(\frac{n_2 \cos[\theta_t]}{n_1 \cos[\theta_0]}\right) \cdot t^2}$$

But this is in terms of four variables: two in the "input" medium (n_1, θ_0) and two in the "output" medium (n_2, θ_t). However, Snell's law gives us another relationship so that we can evaluate this expression based on variables in the "input" medium only:

$$n_1 \sin\left[\theta_0\right] = n_2 \sin\left[\theta_t\right] \implies \sin\left[\theta_t\right] = \frac{n_1}{n_2}\sin\left[\theta_0\right]$$

$$\implies \cos\left[\theta_t\right] = \sqrt{1 - \sin^2\left[\theta_t\right]} = \sqrt{1 - \left(\frac{n_1}{n_2}\sin\left[\theta_0\right]\right)^2}$$

Thus we can write down the transmittance T in terms of the refractive indices and the incident angle:

$$T = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2\left[\theta_0\right]}}{n_1 \cos\left[\theta_0\right]}\right) \cdot t^2$$

6. The index of refraction can be approximately represented by Cauchy's equation:

$$n \cong A + \frac{B}{\lambda_0^2}$$

where λ_0 is the wavelength in vacuum. For a particular material at $\lambda_0 = 500\,\text{nm}$, the coefficients are:

$$\begin{aligned} A &= 1.5 \\ B &= 3 \cdot 10^4\,\text{nm}^2 \end{aligned}$$

Find the phase and modulation ("group") velocities of light in this material at this wavelength.

SOLUTION:

$$n = 1.5 + \frac{3 \cdot 10^4\,\text{nm}^2}{(500\,\text{nm})^2} = 1.62$$

$$\boxed{\text{v}_\phi = \frac{c}{n} = \frac{c}{1.62} \cong 1.85 \times 10^8\,\frac{\text{m}}{\text{sec}} = \frac{\omega}{k}}$$

$3 \times 10^8 \div \left(1.5 + \frac{3 \cdot 10^4\,\text{nm}^2}{(500\,\text{nm})^2}\right) = 1.851\,9 \times 10^8$

$$\begin{aligned} \omega &= \frac{ck}{n} \Longrightarrow \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{ck}{n}\right) = \frac{c}{n} + ck \cdot \frac{d}{dk}\left(\frac{1}{n}\right) \\ &= \frac{c}{n} + ck \cdot \left(-\frac{1}{n^2}\frac{dn}{dk}\right) \\ &= \frac{c}{n} - \frac{c}{n} \cdot \left(\frac{k}{n}\frac{dn}{dk}\right) \\ &= \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right) \\ \text{v}_{\text{mod}} &= \text{v}_\phi \left(1 - \frac{k}{n}\frac{dn}{dk}\right) \end{aligned}$$

So $\text{v}_\phi > \text{v}_{\text{mod}}$ *(normal dispersion) exists if* $\frac{dn}{dk} > 0$, *which means that* $\frac{dn}{d\lambda} < 0$. *We can convert this to an expression in terms of* λ_0 *by appropriate substitution:*

$$\begin{aligned} k &= \frac{2\pi}{\lambda_0} \\ \frac{dn}{dk} &= \frac{dn}{d\lambda_0} \cdot \frac{d\lambda_0}{dk} = \frac{d\lambda_0}{dk} \cdot \frac{dn}{d\lambda_0} \end{aligned}$$

$$
\begin{aligned}
\mathrm{v}_{\mathrm{mod}} &= \mathrm{v}_\phi \left(1 - \frac{2\pi}{n\lambda_0}\left(\frac{d\lambda_0}{dk}\right)\left(\frac{dn}{d\lambda_0}\right)\right) \\
&= \mathrm{v}_\phi \left(1 - \frac{2\pi}{n\lambda_0}\left(-\frac{\lambda_0^2}{2\pi}\right)\left(\frac{dn}{d\lambda_0}\right)\right) \\
&= \mathrm{v}_\phi \left(1 + \frac{\lambda_0}{n}\left(\frac{dn}{d\lambda_0}\right)\right) \\
&= \mathrm{v}_\phi \left(1 + \frac{\lambda_0}{n}\frac{d}{d\lambda_0}\left(1.5 + \frac{3\times 10^{+4}}{\lambda_0^2}\right)\right) \\
&= \mathrm{v}_\phi \left(1 + \frac{\lambda_0}{n}\frac{d}{d\lambda_0}\cdot\left(-2\lambda_0^{-3}\cdot 3\times 10^{+4}\right)\right) \\
&= \mathrm{v}_\phi \left(1 + \frac{\lambda_0}{n}\frac{d}{d\lambda_0}\cdot\left(-2\lambda_0^{-3}\cdot 3\times 10^{+4}\right)\right) \\
&= \mathrm{v}_\phi \left(1 - \frac{6\times 10^{+4}}{n\lambda_0^2}\right)
\end{aligned}
$$

At $\lambda_0 = 500\,\mathrm{nm}$

$$
\boxed{\mathrm{v}_{\mathrm{mod}} = \mathrm{v}_\phi \left(1 - \tfrac{6\times10^{+4}}{1.62\cdot 500^2}\right) \simeq 0.85\mathrm{v}_\phi \cong 1.57\times 10^8\,\tfrac{\mathrm{m}}{\mathrm{sec}}}
$$

$$
\mathrm{v}_{\mathrm{mod}} < \mathrm{v}_\phi \Longrightarrow \boxed{\textit{normal dispersion}}
$$

You could have inferred that the dispersion is normal from the expression for n*, and you received partial credit if you did that without calculating* $\mathrm{v}_{\mathrm{mod}}$.

Statistics: $\mu \cong 58.5, \sigma \cong 9, \max = 82, \min = 49$