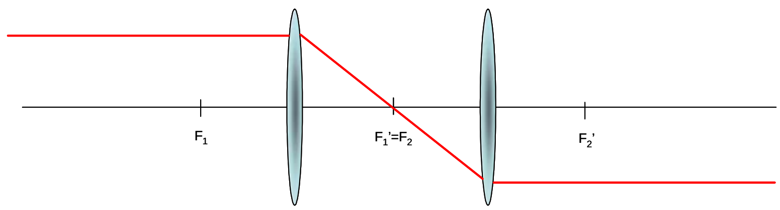
SIMG-712-01 FINAL EXAM Solutions

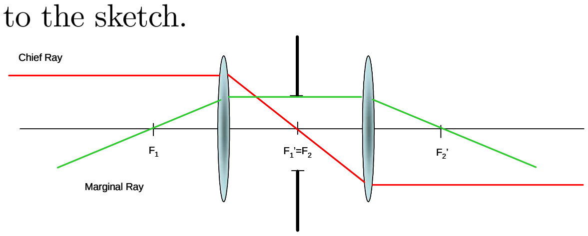
There are 9 problems with point values listed.
Select Problems whose values equal or exceed 100%: your score will be the ratio of points received to points attempted (note optional extra-credit question at end!)
Standard Hint: make sketches before writing down equations.
Show your work!! State any assumptions you make.
Possibly useful expressions listed at end.
First impressions can deceive! If the problem looks difficult, it may not be; if it looks easy, be careful!

1. (25%) You have two identical equiconvex lenses available; each has diameter d and focal length $f > 0$. The system is to be used in light with wavelength λ_0.

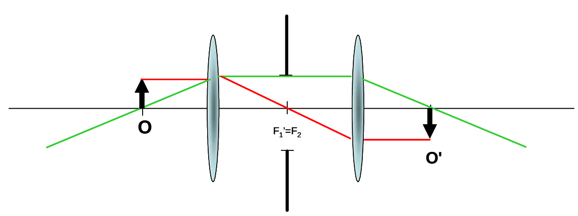
   (a) Sketch the afocal system that can be constructed from these two lenses.
   "Afocal" $\Longrightarrow$ "no focal points," or the focal points are at infinity. In short, this is a telescope.



   (b) A zero-power aperture of diameter $\frac{d}{2}$ is placed at the plane halfway between the two lenses in the system sketched in (a). If the object is a point source located at ∞, determine which element is the stop and use this information to add the marginal and chief rays to the sketch.



   (c) Now consider a planar object located at a distance $s_1 = f$ "in front" of the first lens and oriented perpendicular to the optical axis. Locate and characterize the image.



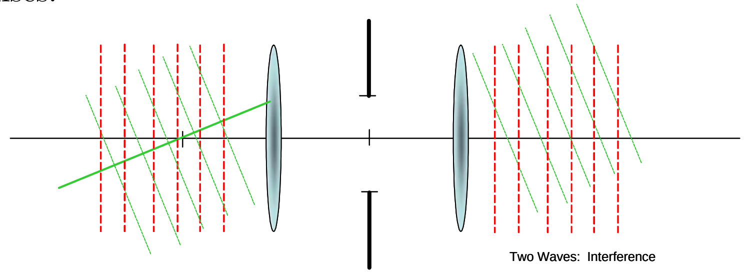
(d) A small opaque spot is place in the system exactly at the center of the zero-power aperture between the two lenses. Explain what happens to the image created by the system of the "on-axis" point source located at ∞ from part (b). Describe the effect of varying the diameter of this small spot on the observed output.

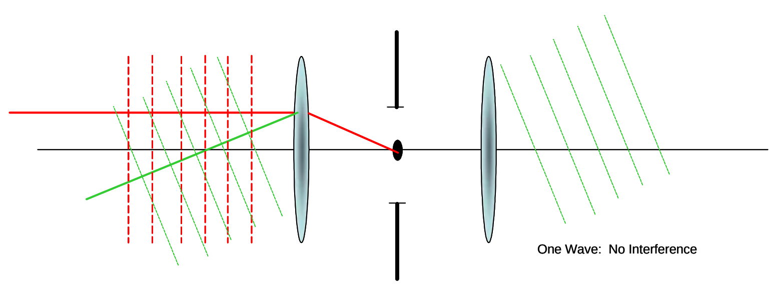
In a geometrical optics model, this opaque spot "blocks" rays from an on-axis object at infinity. In a wave optics model, the diffraction "spot" from the object at infinity is the squared magnitude of the Fourier transform of the aperture function of the first lens. If circular with diameter d, the diffracted light pattern is:

$$\mathcal{F}_2\left\{CYL\left(\frac{r}{d}\right)\right\} = \frac{\pi d^2}{4} SOMB\,(d\rho) \propto SOMB\left(r\frac{d}{\lambda f}\right)$$

If the size of the opaque spot is smaller than the diffraction pattern, then a significant amount of light will reach the output; if the opaque spot is smaller, then little light will reach the output.

(e) A second point source that emits the same wavelength λ_0 is added at an infinite distance from the first lens but that is "off axis," so that the wavefronts from the second source are "tilted" relative to the optical axis. Describe the output created by the system in two cases: before and after the small opaque spot is inserted between the two lenses.





2. (25%) You have been given the job of designing an imaging system using a CCD sensor whose square pixels are $10\,\mu\text{m} \times 10\,\mu\text{m}$. Assume that there are no "gaps" between the pixels so that all photons that reach the sensor are imaged. The diameter of the optic is d and the focal length is f.

   (a) Determine the f/# of the system such that the diameter of the central core of the diffraction spot (the circle enclosed by the first zero of the Fraunhofer diffraction pattern) is fully enclosed in a CCD pixel; assume that $\lambda_0 = 550\,\text{nm}$.

   Use the diffraction formula in the Fraunhofer region. The stop of the system is assumed to be circular (since the lens is assumed to be circular).

$$\begin{aligned} D &\cong 2.44\frac{\lambda f}{d} = 2.44 \cdot \lambda \cdot \text{f/\#} \\ &\Longrightarrow \text{f/\#} \cong \frac{D}{2.44\lambda} = \frac{10\,\mu\text{m}}{2.44 \cdot 550\,\text{nm}} = \frac{10000}{1342} \\ &\boxed{7.5 \cong \text{f/\#}} \end{aligned}$$

   (b) Determine the focal length of the system that results in an "angular resolution" (also called a "plate scale") of 1 arcsecond per pixel at the same wavelength.

$$\begin{aligned} 1 \text{ arcsecond} &= \frac{1}{60} \text{ arcminute} = \frac{1}{3600}^{\circ} = \frac{1}{3600}^{\circ} \cdot \frac{\pi}{180} \frac{\text{radians}}{^{\circ}} \\ &\cong 4.85 \cdot 10^{-6} \text{ radians} = 4.85\ \mu\text{radians} \end{aligned}$$

$$\begin{aligned} f\,[\mu\text{m}] \cdot 4.85 \cdot 10^{-6} \text{ radians} &= 10\,\mu\text{m} \\ \Longrightarrow f = \frac{10\,\mu\text{m}}{4.85 \cdot 10^{-6}} &\cong 2062\,\text{mm} \\ \boxed{f \cong 2.062\,\text{m} \cong 81.2\,\text{in}} \end{aligned}$$

   (c) Use the result of part (b) to determine the diameter of the optic that also satisfies the condition in part (a).

$$\text{f/7.5} \Longrightarrow d = \frac{f}{\text{f/\#}} \Longrightarrow d \cong \frac{2062\,\text{mm}}{7.5}$$

$$\boxed{277.4\,\text{mm} \cong 10.9\,\text{in} \cong d}$$

3. (25%) Two point sources that emit the same wavelength λ_0 are located at $[x, y, z] = \left[\pm\frac{d}{2}, 0, 0\right]$. The light from these sources is observed at an observation plane parallel to the $x-y$ plane and centered at $[0, 0, z_1]$, where z_1 is sufficiently large that the wavefronts from the sources may be modeled as planes. The waves are added to create an interference pattern.

   (a) Describe and sketch the irradiance pattern at the observation plane. You don't have to "derive" the equation for the pattern, but your description and your graph should be quantitative.

   This is two-wave interference, so the relationship applies:

$$L\lambda \cong dD$$
$$\Longrightarrow D = \frac{L\lambda}{d} \to \frac{z_1\lambda}{d}$$

   so the irradiance pattern is:

$$|f[x, y; z_1]|^2 \propto \frac{1}{2}\left(1 + \cos\left[2\pi\frac{x}{D}\right]\right) \cdot 1[y] = \frac{1}{2}\left(1 + \cos\left[2\pi\frac{x}{\left(\frac{z_1\lambda}{d}\right)}\right]\right) \cdot 1[y]$$

   (b) The irradiance at the observation plane is recorded on a photographic emulsion so that the transmission $t[x, y]$ of the emulsion is identical to the normalized"complement" of the incident irradiance, i.e., the transmittance is "0" where the irradiance is its maximum and "1" where the irradiance is zero. In other words, the transmittance pattern of the developed film is the "negative" of the irradiance. This recording process is assumed to be exactly linear. Sketch the transmittance $t[x, y]$ function of this "transparency."

$$t[x, y; z_1] = 1 - \frac{|f[x, y; z_1]|^2}{|f|^2_{\max}} = \frac{1}{2}\left(1 - \cos\left[2\pi\frac{x}{\left(\frac{z_1\lambda}{d}\right)}\right]\right) \cdot 1[y]$$

   (c) The developed film with the transmittance pattern $t[x, y]$ is placed in its original location $[0, 0, z_1]$ and illuminated with light from a single on-axis point source at the origin of coordinates $[x, y, z] = [0, 0, 0]$. The light that passes through the transparency then propagates a large distance z_2 into the Fraunhofer diffraction region (so that the new observation plane is located at $[0, 0, z_1 + z_2]$. Sketch and describe the irradiance pattern that is observed at the new observation plane. You may ignore any constant terms; just show the functional form of the measured irradiance.

   Since the distance is sufficient for Fraunhofer diffraction, the output pattern is proportional to the squared magnitude of the Fourier transform of $t[x, y; z_1]$ *evaluated at* $\xi = \frac{x}{\lambda z_2}, \eta = \frac{y}{\lambda z_2}$:

$$
\begin{aligned}
T[\xi,\eta] &= \mathcal{F}_2\{f[x,y]\} = \frac{1}{2}\delta[\xi] - \frac{1}{4}\left(\delta\left[\xi + \frac{1}{\left(\frac{z_1\lambda}{d}\right)}\right] + \delta\left[\xi - \frac{1}{\left(\frac{z_1\lambda}{d}\right)}\right]\right) \\
&= \frac{1}{2}\delta[\xi,\eta] - \frac{1}{4}\left(\delta\left[\xi + \frac{d}{\lambda z_1},\eta\right] + \delta\left[\xi - \frac{d}{\lambda z_1},\eta\right]\right) \\
T\left[\frac{x}{\lambda z_2},\frac{y}{\lambda z_2}\right] &= \frac{1}{2}\delta\left[\frac{x}{\lambda z_2},\frac{y}{\lambda z_2}\right] - \frac{1}{4}\left(\delta\left[\frac{x}{\lambda z_2} + \frac{d}{\lambda z_1},\frac{y}{\lambda z_2}\right] + \delta\left[\frac{x}{\lambda z_2} - \frac{d}{\lambda z_1},\frac{y}{\lambda z_2}\right]\right) \\
&= \frac{|\lambda z_2|^2}{2}\delta[x,y] + \frac{|\lambda z_2|^2}{4}\left(\delta\left[x + d\frac{z_2}{z_1},y\right] + \delta\left[x - d\frac{z_2}{z_1},y\right]\right)
\end{aligned}
$$

The irradiance is the squared magnitude, which also is composed of three Dirac delta functions:

$$
g[x,y] \propto \delta[x,y] + \frac{1}{16}\left(\delta\left[x + d\frac{z_2}{z_1},y\right] + \delta\left[x - d\frac{z_2}{z_1},y\right]\right)
$$

The distance away from the axis has been scaled by a factor of $\frac{z_2}{z_1}$*, which is analogous to a transverse magnification.*

4. (25%) A digital image $f_q[n,m]$ includes two distinct spatial regions created from random "noise," i.e., unpredictable variations of nonnegative integers. There two regions have the same mean gray values but different variances (i.e., one region is "noisier"); an example is shown. Your mission (should you decide to accept it) is to segment the two regions by applying pixel (point) operators and linear shift-invariant local neighborhood operations (i.e., convolutions). The only nonlinear operations available to you are pixel operations (e.g., thresholding, nonlinear scaling of gray values).

   (a) Sketch the histogram of the original image $f_q[n,m]$.

   A possible histogram is the sum of the histograms of the component regions. The example shown was constructed from uniformly distributed noise (see below)

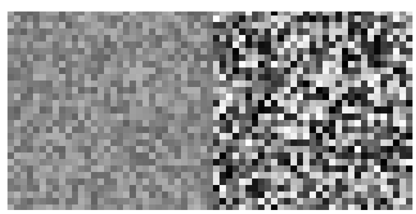
   (b) Specify the sequence of operations to segment the regions. (HINT: you need to convert "noisiness" to "gray level" and then select a threshold). Explicitly define any kernels used in convolutions.

   To convert "noisiness" to "gray level", use a differencing operator (highpass filter). I suggest using a flavor of Laplacian kernel, e.g.,

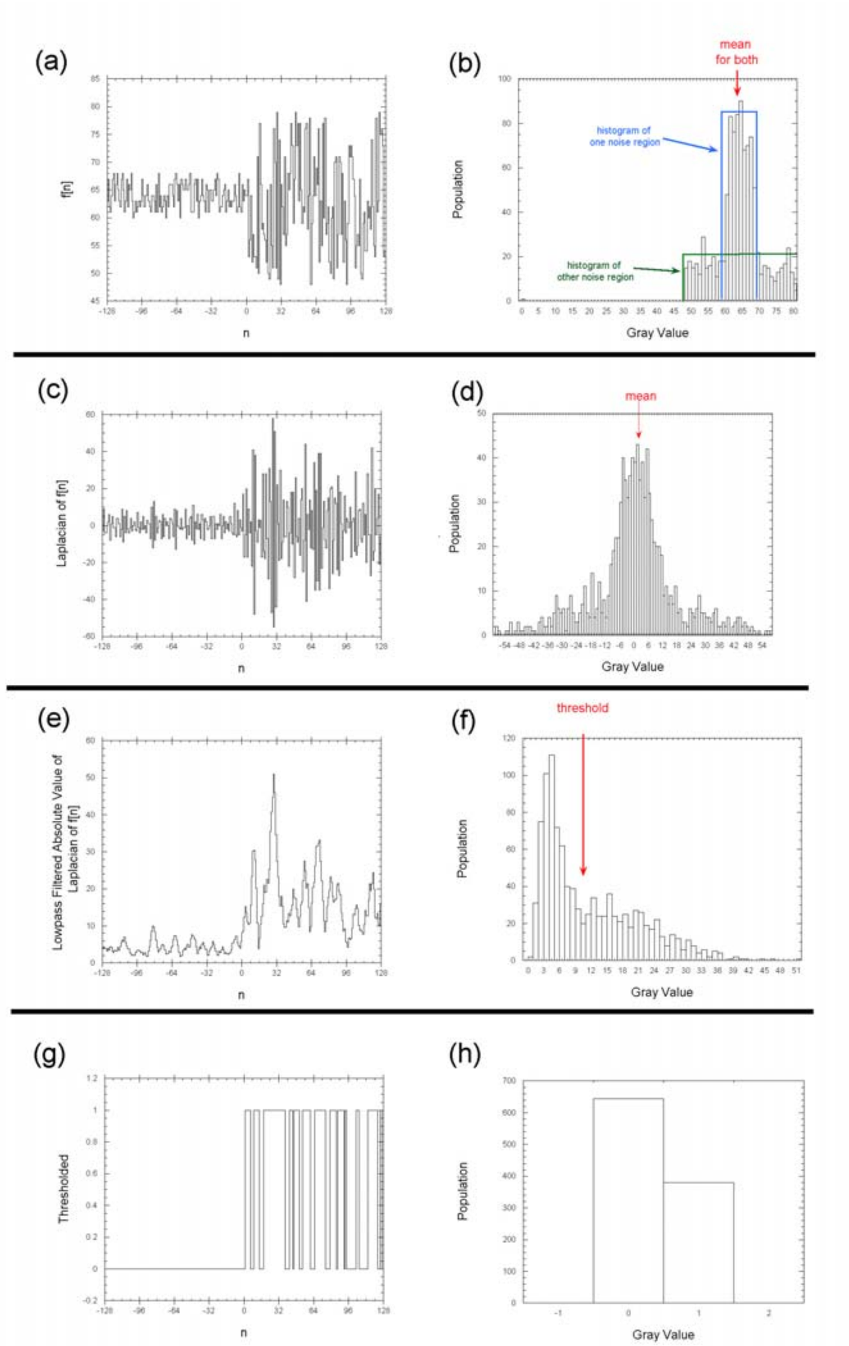
-1	-1	-1
-1	+8	-1
-1	-1	-1

   This blocks the constant part and amplifies the high-frequency noise. But the output is bipolar! So apply a nonlinear point operator of absolute value:

   (c) Sketch approximate histograms that result from each step in your sequence.



Magnified view of section of $f_q[n,m]$ near boundary between two regions of noise with the same mean and different variances.



1-D slices of the image and the corresponding histogram after each step: (a,b) original image; (c,d) after Laplacian; (e,f) after absolute value and 3×3 averaging, (g,h) after thresholding.

5. (15%) Consider the statement: "In normal use, the magnitude of the transverse magnification of an imaging system increases as its equivalent focal length is increased." Find an expression for the transverse magnification in terms of the focal length that demonstrates the truth or falsehood of this statement. State any conditions that must be satisfied in the mathematical expression (e.g., why does the sentence include the caveat "in normal use"?).

We know that the equation for transverse magnification is:

$$M_T = -\frac{s'}{s}$$

where s and s' are measured from the corresponding principal points if this is a lens "system." Now consider the imaging equation for optical systems:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{eff}}$$

Find an expression for s' :

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f_{eff}} \implies s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = \frac{sf}{s-f}$$

Substitute that into the equation for transverse magnification:

$$M_T = -\frac{s'}{s} = -\frac{\left(\frac{sf}{s-f}\right)}{s} = \frac{f}{f-s}$$

We're almost there, but this has two instances of the focal length f. Extract the common factor to obtain:

$$M_T = \frac{f}{f-s} = \frac{f}{s}\left(\frac{1}{\frac{f}{s}-1}\right) = -\frac{f}{s}\left(\frac{1}{1-\frac{f}{s}}\right)$$

Now, if $f < s$, as is the case "in normal use" (the object is farther away than the focal length so that the image is real), we can use the formula given in the "useful formulas":

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \text{ if } |t| < 1$$

$$\implies M_T = -\frac{f}{s}\left(\frac{1}{1-\frac{f}{s}}\right) = -\frac{f}{s}\left(1 + \frac{f}{s} + \left(\frac{f}{s}\right)^2 + \cdots\right) = -\sum_{n=1}^{\infty}\left(\frac{f}{s}\right)^n$$

If $f << s$ (as also is often the case, particularly for remote-sensing applications), then we can truncate the series at some small number of terms without incurring much error:

$$M_T = -\frac{f}{s}\left(1 + \frac{f}{s} + \left(\frac{f}{s}\right)^2 + \cdots\right) \cong -\frac{f}{s} - \frac{f^2}{s^2}$$

which clearly increases in absolute value with increasing f.

6. (15%) Histogram equalization:

(a) A discrete image $f_q[n,m]$, where f_q is an integer in the range $0 \leq f_q \leq 2^m - 1$ is processed to create the image $g_q[n,m]$ with a "flat" histogram; $g_q[n,m]$ is then subjected to the same "flattening" operation to produce $p_q[n,m]$. Describe the similarities and differences among the three images.

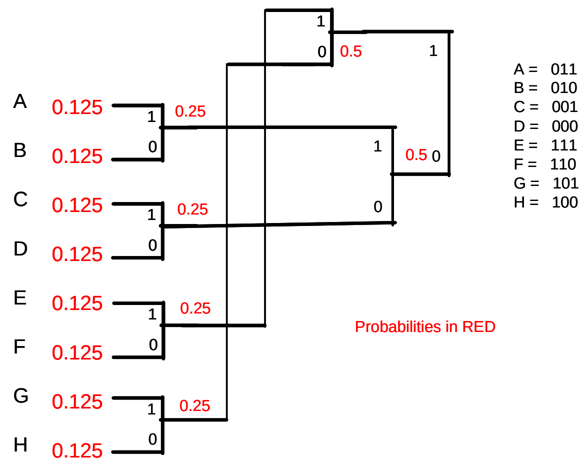
Since the cumulative histogram of the second image $g_q[n,m]$ *is the lookup table for the second transformation, and since that cumulative already is as close to the identity lookup table as possible, the third image* $p_q[n,m]$ *is identical to the second image* $g_q[n,m]$. *Both (likely) will have more amplitude at large spatial frequences that* $f_q[n,m]$ *because of the nonlinear lookup table.*

(b) Comment on the value and the problems with the use of histogram equalization to compare images of the same scene taken under different conditions.

Since equalization is generally nonlinear, its impact on the spatial frequency content is unpredictable (except for the fact that the support of the frequency spectrum will increase). Therefore, it is dangerous to try to compare images whose histograms have been equalized.

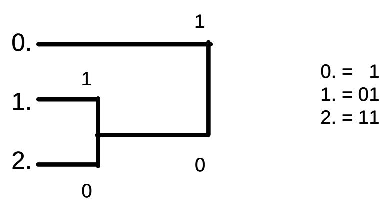
7. (15%) Information Content:

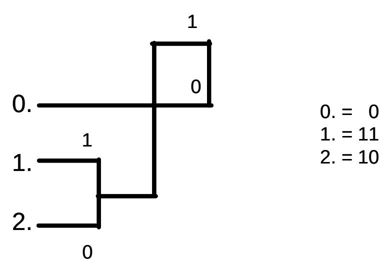
(a) Derive the Huffman code for a 3-bit image with a flat histogram. (SHOW YOUR WORK!)



(b) The gray values of an image are f_q where $0 \leq f_q \leq 2$ and the probability of each level ($p\left[f_q\right] \leq 1$) is not know. How many unique Huffman codes exist for this image?

The possible Huffman codes are:





The two codes are complements of each other.

8. (10%) Classify the action of the following kernels (e.g., "highpass," "lowpass") and specify an application for each (all locations not shown have value "0").

(a)

0	0	0
−1	+1	0
0	0	0

The sum of the weights is zero, which demonstrates that this kernel will "block" constant terms. It is the "negative" of the first derivative operator in the x-direction. It is a "highboost" filter since it amplifies large spatial frequencies. Useful perhaps as an edge detector.

(b)

0	0	$-\frac{1}{2}$
0	+2	0
$-\frac{1}{2}$	0	0

The sum of the weights is one, which shows that this operator will "pass" 3×3 *regions with constant gray values unchanged.*

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & +2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & +1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is the sum of the identity operator and an operator that is proportional to the second derivative. This will pass low-frequency components and enhance high-frequency ones. It is a "directional sharpener."

(c)

+1	0	−1	0	+1	0	−1	0	+1	0	−1

The sum of the weights is zero; blocks constants. This impulse response includes (almost) three cycles of a sinusoid that oscillates with a period of 4 pixels. It will attenuate other patterns (e.g., if convolved with a "checkerboard" pattern that oscillates at the Nyquist rate, the output will be zero). So this is a "bandpass" filter.

(d)

+2	−1	−1
−1	+2	−1
−1	−1	+2

The sum of the weights is zero, so this blocks constants. It "averages" along the diagonal (from 10:30 to 4:30) and differences across the orthogonal diagonal. This could be used as a "line detector" if the line is one pixel wide oriented along that diagonal.

9. (10%) The gradient operator $\nabla f[n,m]$ produces a vector at each pixel. The magnitude of the gradient is defined to be the square root of the sum of the squares of the vector components, and is sometimes approximated as the sum of the absolute values of the vector components:

$$\begin{aligned} |\boldsymbol{\nabla} f[n,m]| &= \sqrt{\left[(\nabla f[n,m])_x\right]^2 + \left[(\nabla f[n,m])_y\right]^2} \\ &\cong |(\nabla f[n,m])_x| + \left|(\nabla f[n,m])_y\right| \end{aligned}$$

The azimuth angle of the gradient is defined as:

$$\Phi\{\nabla f[n,m]\} = \tan^{-1}\left[\frac{(\nabla f[n,m])_y}{(\nabla f[n,m])_x}\right]$$

(a) Evaluate these two statements for the magnitude of the gradient of a binary (or *bitonal*) image $f[n,m]$ of size $N \times N$ pixels that contains a rectangular region of level "1" surrounded by pixels with level "0."

We need to calculate the gradient for horizontal edges, vertical edges, and "corners".

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$$\Longrightarrow \quad \frac{\partial f}{\partial x} = f[n+1,m] - f[n,m] =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	+1	0	0	0	-1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$$\Longrightarrow \quad \frac{\partial f}{\partial y} = f[n,m+1] - f[n,m] =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	-1	-1	-1	-1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

The "first" definition of the magnitude of the gradient is:

$$\sqrt{(\partial_x * f[n,m])^2 + (\partial_y * f[n,m])^2} \quad =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	+1	+1	+1	$+\sqrt{2}$	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

$$|\partial_x * f[n,m]| + |\partial_y * f[n,m]| \quad =$$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	+1	+1	+1	+1	+2	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	+1	0	0	0	+1	0	0
0	0	+1	+1	+1	+1	0	0
0	0	0	0	0	0	0	0

(b) What are the possible edge "directions" that would be evaluated for this image?

$=$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	**0**	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{\pi}{2}$	$-\frac{3\pi}{4}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	**0**	0	0	0	$-\boldsymbol{\pi}$	0	0
0	0	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	$+\frac{\pi}{2}$	0	0
0	0	0	0	0	0	0	0

Statistics without bonus:

$\mu = 72.59$, $\sigma = 10.56$, max $= 88$, min $= 45$