SIMG-455 Midterm Exam Solutions

1. We used the Argand diagram (also called the phasor diagram) to represent temporal oscillatory motion.

   (a) Use the Argand diagram to demonstrate that the superposition of three oscillations with identical frequencies is an oscillation with that frequency.

   An oscillation with angular temporal frequency ω_0 is represented on an Argand diagram as a vector with length equal to the amplitude of the oscillation that rotates around the origin at frequency $\nu_0 = \frac{\omega_0}{2\pi}$ Hertz. This is true of all such oscillations. The superposition of any number (not just three) oscillations with frequency ω_0 evaluated at $t = 0$ yields the vector sum of these oscillations. Since all of the component vectors rotate with frequency ν_0, so must the sum. In short, the sum of oscillations with frequency ν_0 yields an oscillation with frequency ν_0, regardless of the component amplitudes.

   (b) Use the Argand diagram to find the sum of these three oscillations:

$$\begin{aligned} f_1[t] &= \cos\left[2\pi\nu_0 t + \frac{\pi}{3}\right] \\ f_2[t] &= \cos\left[2\pi\nu_0 t - \frac{\pi}{3}\right] \\ f_3[t] &= -\cos[2\pi\nu_0 t] \end{aligned}$$

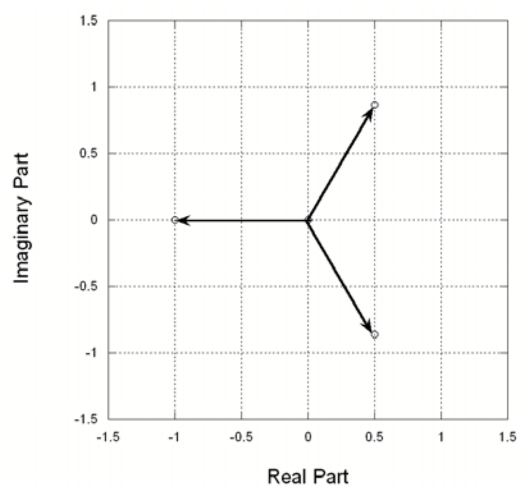
   where ν_0 is arbitrary.

   Note that:

$$-\cos[2\pi\nu_0 t] = \cos[2\pi\nu_0 t + \pi]$$

   Plot these on an Argand diagram so that:

$$\begin{aligned} f_1[t=0] &= \text{Re}\left\{\exp\left[+i\left(2\pi\nu_0 \cdot 0 + \frac{\pi}{3}\right)\right]\right\} = \exp\left[+i\frac{\pi}{3}\right] = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ f_2[t] &= \text{Re}\left\{\exp\left[+i\left(2\pi\nu_0 \cdot 0 - \frac{\pi}{3}\right)\right]\right\} = \exp\left[-i\frac{\pi}{3}\right] = \frac{1}{2} - i\frac{\sqrt{3}}{2} \\ f_3[t] &= \text{Re}\{\exp[+i(2\pi\nu_0 \cdot 0 - \pi)]\} = \exp[+i0] = 1 + 0i \end{aligned}$$



The sum of these at $t = 0$ is:

$$f_1\left[t=0\right]+f_2\left[t=0\right]+f_2\left[t=0\right]=\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)+(-1+i\cdot 0)=0$$

So the summation oscillation has amplitude 0.

2. A function $f_1[z,t]$ is created by summing two sinusoidal functions

$$f_1[z,t] = 2\cos\left[-2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2}\right] - 2\cos\left[2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2}\right]$$

Find an expression for the result as the *product* of two sinusoidal functions.

Recall that:

$$\cos A + \cos B = 2\cos\left[\frac{A+B}{2}\right]\cdot\cos\left[\frac{A-B}{2}\right]$$

You can use this directly (if you remember it) or you can derive it as we did in class. I'll do that below for completeness:

$$\begin{aligned}\cos A + \cos B &= \mathrm{Re}\left\{e^{+iA} + e^{+iB}\right\} = \mathrm{Re}\left\{e^{+i\frac{A}{2}}e^{+i\frac{A}{2}} + e^{+i\frac{B}{2}}e^{+i\frac{B}{2}}\right\} \\ &= \mathrm{Re}\left\{e^{+i\frac{A}{2}}e^{+i\frac{A}{2}}\left(e^{+i\frac{B}{2}}e^{-i\frac{B}{2}}\right) + e^{+i\frac{B}{2}}e^{+i\frac{B}{2}}\left(e^{+i\frac{A}{2}}e^{-i\frac{A}{2}}\right)\right\}\end{aligned}$$

because $e^{+i\frac{A}{2}}e^{-i\frac{A}{2}} = e^{i\cdot 0} = 1$. *Now continue by rearranging:*

$$\begin{aligned}&\mathrm{Re}\left\{e^{+i\frac{A}{2}}e^{+i\frac{A}{2}}\left(e^{+i\frac{B}{2}}e^{-i\frac{B}{2}}\right) + e^{+i\frac{B}{2}}e^{+i\frac{B}{2}}\left(e^{+i\frac{A}{2}}e^{-i\frac{A}{2}}\right)\right\} \\ =\;& \mathrm{Re}\left\{\left(e^{+i\frac{A}{2}}e^{+i\frac{B}{2}}\right)\left(e^{+i\frac{A}{2}}e^{-i\frac{B}{2}}\right) + \left(e^{+i\frac{A}{2}}e^{+i\frac{B}{2}}\right)\left(e^{-i\frac{A}{2}}e^{+i\frac{B}{2}}\right)\right\} \\ =\;& \mathrm{Re}\left\{\left(e^{+i\frac{A+B}{2}}\right)\left(e^{+i\frac{A-B}{2}} + e^{-i\frac{A-B}{2}}\right)\right\} \\ =\;& \mathrm{Re}\left\{\left(e^{+i\frac{A+B}{2}}\right)\left(2\cos\left[\frac{A-B}{2}\right]\right)\right\} \\ =\;& 2\cos\left[\frac{A-B}{2}\right]\cdot\mathrm{Re}\left\{e^{+i\frac{A+B}{2}}\right\} \\ =\;& 2\cos\left[\frac{A-B}{2}\right]\cdot\cos\left[\frac{A+B}{2}\right]\end{aligned}$$

So with that in hand, we can rewrite the sum of the two traveling-wave oscillations:

$$\begin{aligned}&2\cos\left[-2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2}\right] - 2\cos\left[2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2}\right] \\ &= 2\left(\cos\left[-2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2}\right] + \cos\left[2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t - \frac{\pi}{2} + \pi\right]\right) \\ &= 2\left(\cos\left[-\left(2\pi\frac{z}{\lambda_0} + 2\pi\nu_0 t + \frac{\pi}{2}\right)\right] + \cos\left[2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t + \frac{\pi}{2}\right]\right) \\ &= 2\left(\cos\left[2\pi\frac{z}{\lambda_0} + 2\pi\nu_0 t + \frac{\pi}{2}\right] + \cos\left[2\pi\frac{z}{\lambda_0} - 2\pi\nu_0 t + \frac{\pi}{2}\right]\right)\end{aligned}$$

$$= 2 \cdot 2 \cdot \cos\left[\frac{2\pi\frac{z}{\lambda_0} + 2\pi\frac{z}{\lambda_0}}{2} + \frac{2\pi\nu_0 t - 2\pi\nu_0 t}{2} + \frac{\frac{\pi}{2} + \frac{\pi}{2}}{2}\right]$$

$$\cdot \cos\left[\frac{2\pi\frac{z}{\lambda_0} - 2\pi\frac{z}{\lambda_0}}{2} + \frac{2\pi\nu_0 t - (-2\pi\nu_0 t)}{2} + \frac{\frac{\pi}{2} - \frac{\pi}{2}}{2}\right]$$

$$= 4 \cdot \cos\left[2\pi\frac{z}{\lambda_0} + 0 + \frac{\pi}{2}\right] \cdot \cos\left[0 + 2\pi\nu_0 t + 0\right]$$

$$\boxed{f_1[z,t] = 4 \cdot \cos\left[2\pi\frac{z}{\lambda_0} + \frac{\pi}{2}\right] \cdot \cos\left[2\pi\nu_0 t\right]} \text{ \textit{standing wave}}$$

$$= -4 \cdot \sin\left[2\pi\frac{z}{\lambda_0}\right] \cdot \cos\left[2\pi\nu_0 t\right] \text{ \textit{if you want to go this far.}}$$

3. Consider four ideal linear polarizers oriented at $\theta_1 = 0°$, $\theta_2 = 30°$, $\theta_3 = 60°$, and $\theta_4 = 90°$. Natural light with "intensity" I_0 (proportional to E_0^2) is incident on the first polarizer such that the emerging irradiance is I_1.

   (a) Specify the state of polarization of the light emerging from the last linear polarizer.

   Since the last element is a linear polarizer oriented at $\theta_4 = 90°$*, the light emerging (if any) MUST BE linearly polarized at* $\theta_4 = 90°$

   (b) Find the "intensity" or "irradiance" emerging from the last polarizer in terms of I_0; you may use any method, but extra credit will be given for use of Jones vectors and matrices.

   Using Jones vectors, the light emerging from the first polarizer has the form:

$$\mathcal{E}_1 = \frac{I_0}{2} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Recall that the Jones vector for light that is linearly polarized at azimuth angle θ *is:*

$$\mathcal{E}_\theta = I \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

The matrix for a polarizer at angle θ *may be derived easily enough. We need two vector equations to evaluate the four unknown elements of the matrix. If the matrix represents a linear polarizer at angle* θ*, then it must transmit that Jones vector:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and block the Jones vector for light that is linearly polarized at angle $\theta + \frac{\pi}{2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\left[\theta + \frac{\pi}{2}\right] \\ \sin\left[\theta + \frac{\pi}{2}\right] \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Combine into one matrix equation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \left[\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right] = \left[\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right]$$

$$\Longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix}$$

Now solve for the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

So the matrices for the different angles are:

$$\begin{aligned}
\underline{\mathcal{M}}_{30^\circ} &= \begin{bmatrix} \cos^2\frac{\pi}{6} & \cos\frac{\pi}{6}\sin\frac{\pi}{6} \\ \cos\frac{\pi}{6}\sin\frac{\pi}{6} & \sin^2\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \left(\frac{\sqrt{3}}{2}\right)^2 & \frac{\sqrt{3}}{2}\cdot\frac{1}{2} \\ \frac{\sqrt{3}}{2}\cdot\frac{1}{2} & \left(\frac{1}{2}\right)^2 \end{bmatrix} = \frac{1}{4}\cdot\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \\
\underline{\mathcal{M}}_{60^\circ} &= \begin{bmatrix} \cos^2\frac{\pi}{3} & \cos\frac{\pi}{3}\sin\frac{\pi}{3} \\ \cos\frac{\pi}{3}\sin\frac{\pi}{3} & \sin^2\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & \frac{1}{2}\cdot\frac{\sqrt{3}}{2} \\ \frac{1}{2}\cdot\frac{\sqrt{3}}{2} & \left(\frac{\sqrt{3}}{2}\right)^2 \end{bmatrix} = \frac{1}{4}\cdot\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} \\
\underline{\mathcal{M}}_{90^\circ} &= \begin{bmatrix} \cos^2\frac{\pi}{2} & \cos\frac{\pi}{2}\sin\frac{\pi}{2} \\ \cos\frac{\pi}{2}\sin\frac{\pi}{2} & \sin^2\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Now we can string the matrices together with the first one acting first:

$$\begin{aligned}
\underline{\mathcal{M}} &= \underline{\mathcal{M}}_{90^\circ}\cdot\underline{\mathcal{M}}_{60^\circ}\cdot\underline{\mathcal{M}}_{30^\circ} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\cdot\frac{1}{4}\cdot\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}\cdot\frac{1}{4}\cdot\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \frac{3}{8}\begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix}
\end{aligned}$$

Now apply to the light emerging from the first polarizer:

$$\underline{\mathcal{M}}\mathcal{E}_1 = \frac{3}{8}\begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix}\cdot\frac{I_0}{2}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(\frac{3\sqrt{3}}{16}I_0\right)\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Linearly polarized light along the y direction with intensity $\frac{3\sqrt{3}}{16}I_0 \cong 0.325 I_0$

4. :From observations of snowflakes, you might surmise that frozen water has a crystalline structure in a hexagonal form (due to the dominant angular position of the hydrogen atoms at 120°). This anistotropic structure produces different forces on the charges in the molecules, which results in slightly different refractive indices along orthogonal directions. If the indices of refraction along orthogonal directions happen to be $n_1 = 1.313$ and $n_2 = 1.309$, determine the thickness d of ice that would produce a relative phase delay of $\Delta\phi = \frac{\pi}{2}$ for the polarizations along the two directions.

The phase delay through thickness d is determined by the number of wavelengths of light within the medium, which in turn is determined by λ_0 *and* n *:*

$$\begin{aligned} \textit{wavelength in medium with index } n \textit{ is } \lambda &= \frac{\lambda_0}{n} \\ \textit{number of such waves in length } d \textit{ is } N &= \frac{d}{\lambda} = \frac{nd}{\lambda_0} \\ \textit{number of radians in length } d \textit{ is } \Delta\Phi &= 2\pi \cdot \frac{nd}{\lambda_0} \end{aligned}$$

If two different indices n_1 *and* n_2*, the phase differences will differ along the two different axes (i.e., everything is different).*

$$\begin{aligned} \Delta\Phi_1 &= 2\pi \cdot \frac{n_1 d}{\lambda_0} \\ \Delta\Phi_2 &= 2\pi \cdot \frac{n_2 d}{\lambda_0} \\ \Delta\phi &= \Delta\Phi_1 - \Delta\Phi_2 = 2\pi \frac{d}{\lambda_0} (n_1 - n_2) \end{aligned}$$

For a specified $\Delta\phi$*, we can rearrange the equation to find d:*

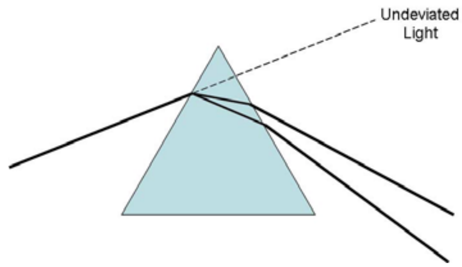
$$d = \Delta\phi \cdot \frac{\lambda_0}{2\pi (n_1 - n_2)}$$

Now substitute the known parameters:

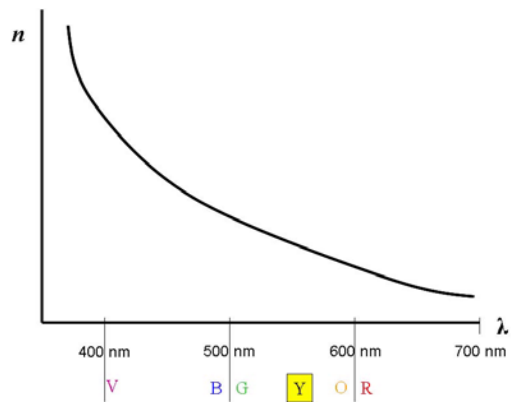
$$d = \frac{\pi}{2} \cdot \frac{500\,\text{nm}}{2\pi (1.313 - 1.309)} = \frac{500\,\text{nm}}{4 (1.313 - 1.309)} = 31,250\,\text{nm} = 31.250\,\mu\text{m} = 0.031\,\text{mm}$$

This is VERY thin, so something must be changed to create a useful quarter-wave plate. Exercise for the student.

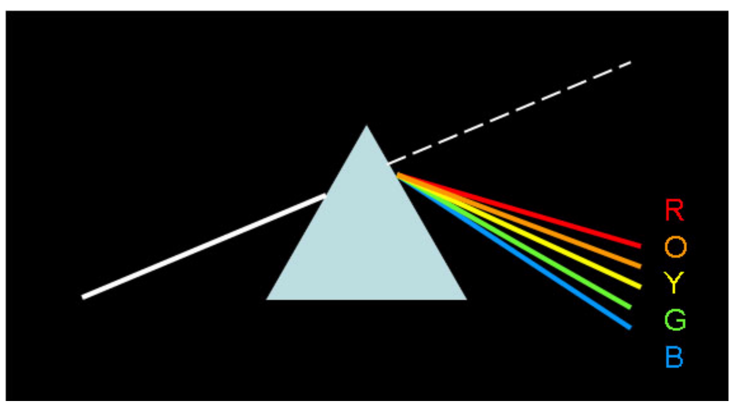
5. Light is dispersed by a glass prism to create the spectrum as shown:



(a) Replicate this sketch on your paper and label the order of the colors that emerge from the prism if the glass exhibits normal dispersion with a curve of this shape (vertical axis of $n[\lambda]$ is arbitrary, but you may assume a reasonable range of values of n)

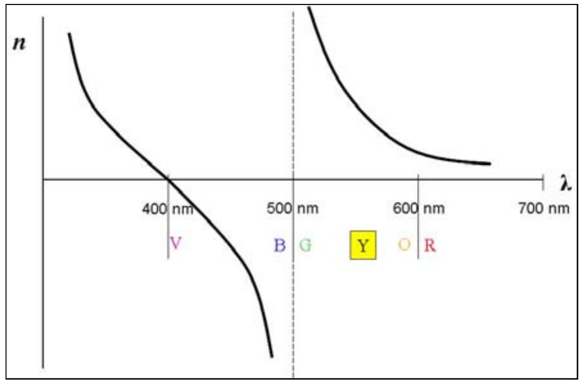


The familiar "ROYGBIV" order of wavelengths; red is deviated the least and blue the most.



(b) Now consider some theoretical substance that exhibits anomalous dispersion with a resonance centered at $\lambda = 500\,\text{nm}$. Use the "undamped" model of the refractive index to sketch a possible graph of $n[\lambda]$ for this material.

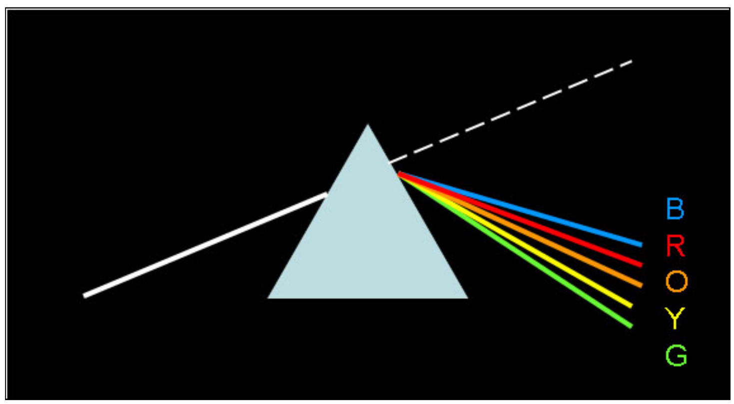
One possible graph of the refractive index near the resonance is:



so the refractive index decreases up to the resonance and then jumps to a much larger value.

(c) Make another sketch of a prism and label the sequence of dispersed colors that would be produced if the prism were made of the material in part (b). Explain any features on your graph.

One possible solution:



The index of refraction for blue light is now much smaller than it was before and the index for red light is now similar to that of blue. The largest index is for wavelengths just longer than 500 nm *(i.e., green and yellow)*