1 Electromagnetic Waves at Interface – Fresnel Equations

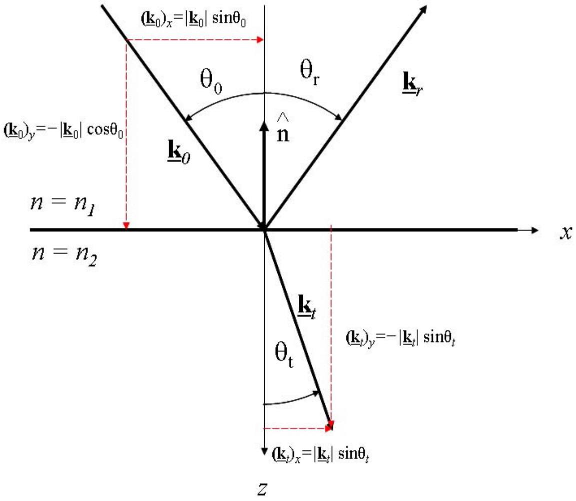
- Beam of light (plane wave) in vacuum or isotropic medium propagates in particular fixed direction specified by Poynting vector until encounters interface with different medium (different refractive index n)
- Light (EM radiation) causes electric charges (electrons) in medium to oscillate
  - Driven electrons in medium emit additional ("scattered") light
  - Scattered waves may travel in any direction (over the sphere of 4π steradians of solid angle)
    - forwards, backwards, sideways, ...
- Oscillating electrons vibrate at frequency of incident light
  - Re-emit scattered light at that frequency
  - If emitted light is "out of phase" with incident light (phase difference $\Delta\Phi \cong \pm\pi$ radians)
    - $\Longrightarrow$ destructive interference $\Longrightarrow$ attenuation
  - Complete attenuation $\Longrightarrow$ absorption
  - Scattered light interferes constructively with incident light in specific directions
    - Reflected and/or Transmitted light
    - Constructive interference of transmitted beam at angle of Snell's law:

$$\theta_{\text{refracted}} = \sin^{-1}\left[\frac{n_1}{n_2}\sin\left[\theta_{\text{incident}}\right]\right]$$

    - Constructive interference of reflection $\Longrightarrow$

$$\theta_{\text{reflected}} = -\theta_{\text{incident}}$$

- Mathematical derivation from:
  1. Maxwell's equations
  2. Boundary conditions at interface
  - three waves ("incident," "reflected," "refracted" = "transmitted")



The $\underline{\mathbf{k}}$ *vectors of the incident, reflected, and "transmitted" (refracted) wave at the interface between two media of index* n_1 *and* n_2 *(where* $n_2 > n_1$ *in the example shown).*

- Derivation leads to:

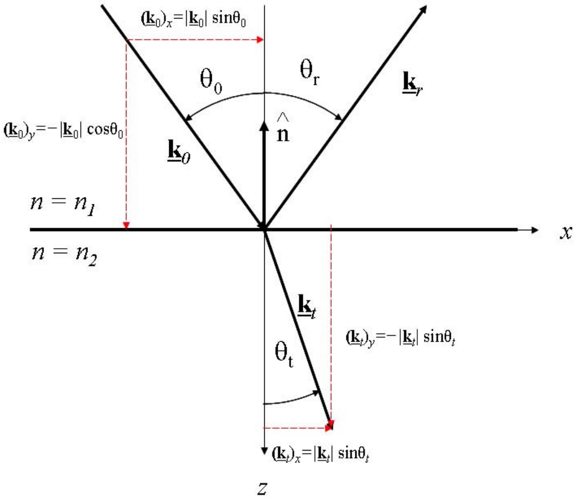
1. Snell's Law that relates incident wave to refracted and transmitted waves;
2. orientations of the electric fields of the three waves (the states of polarization of the three waves), and;
3. relative "strengths" and phases of the three light waves

1.1 Definitions of Vectors

- Consider only plane waves
  - all light (incident, reflected, and transmitted) specified by single wavevectors $\underline{\mathbf{k}}_n$ valid at all points in medium
  - Point in direction of propagation of plane wave

$$|\underline{\mathbf{k}}_n| = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\left(\frac{\lambda_0}{n}\right)} = 2\pi\frac{n}{\lambda_0}$$

-
  - λ_0 = wavelength in vacuum
  - $\lambda_n = \frac{\lambda_0}{n}$ wavelength in medium
  - Interface between media assumed to be $x - y$ plane located at $z = 0$
  - Incident wavevector $\underline{\mathbf{k}}_0$, reflected wavevector $\underline{\mathbf{k}}_r$, transmitted (refracted) wavevector $\underline{\mathbf{k}}_t$ and unit vector $\hat{\underline{\mathbf{n}}}$ normal to the interface:



The $\underline{\mathbf{k}}$ *vectors of the incident, reflected, and "transmitted" (refracted) wave at the interface between two media of index* n_1 *and* n_2 *(where* $n_2 > n_1$ *in the example shown).*

  - Angles θ_0, θ_r, and θ_t measured from normal
  - As drawn:

$$\begin{aligned} \theta_0 &> 0 \\ \theta_t &> 0 \\ \theta_r &< 0 \end{aligned}$$

- Incident and reflected beams are in same medium ($n = n_1$) $\Longrightarrow$ same wavelength λ_1 and same $|\underline{\mathbf{k}}|$:

$$\lambda_1 = \frac{2\pi n_1}{|\underline{\mathbf{k}}_0|} = \frac{2\pi n_1}{|\underline{\mathbf{k}}_r|}$$

$$|\underline{\mathbf{k}}_0| = |\underline{\mathbf{k}}_r| = \frac{\omega_0}{\mathrm{v}_1} = \frac{2\pi n_1}{\lambda_0}$$

- Wavelength of transmitted (refracted) beam differs because $n_2 \neq n_1$:

$$\lambda_2 = \frac{2\pi n_2}{|\underline{\mathbf{k}}_t|}$$

- Normal to interface specified by unit vector perpendicular:

$$\hat{\underline{\mathbf{n}}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

  (n.b., could define $\hat{\underline{\mathbf{n}}}$ in opposite direction, would change signs of angles but have no effect on physics).

- Incident electric field in complex notation:

$$\underline{\mathbf{E}}_{incident} = \underline{\mathbf{E}}_0 \; \exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t + \phi_0\right)\right]$$

  - $\underline{\mathbf{E}}_0$: electric field vector that defines magnitude and direction of polarization
  - $\underline{\mathbf{r}} = [x, y, z]$ is position vector of location where phase $\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t$ measured
  - $\underline{\mathbf{k}}_0 = \left[(k_0)_x \;,\; (k_0)_y \;,\; (k_0)_z\right]$
  - ϕ_0: initial phase of electric field (i.e., evaluated at $\underline{\mathbf{r}} = \underline{\mathbf{0}}, t = 0$)
  - Phases measured at all positions in plane perpendicular to incident wavevector $\underline{\mathbf{k}}_0$ are identical, i.e., incident field is a plane wave

- Reflected and Transmitted waves:

$$\underline{\mathbf{E}}_{\text{reflected}}[\underline{\mathbf{r}}, t] = \underline{\mathbf{E}}_r \; \exp\left[+i\left(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega_r t + \phi_r\right)\right]$$
$$\underline{\mathbf{E}}_{\text{transmitted}}[\underline{\mathbf{r}}, t] = \underline{\mathbf{E}}_t \; \exp\left[+i\left(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega_t t + \phi_t\right)\right]$$

  - where:
    - $\underline{\mathbf{E}}_r$: vector that specifies the magnitude and direction of the reflected electric field
    - ϕ_r: initial phase of reflected field (i.e., evaluated at $\underline{\mathbf{r}} = \underline{\mathbf{0}}, t = 0$)
    - $\underline{\mathbf{E}}_t$: vector that specifies the magnitude and direction of the "transmitted" (refracted) electric field

  - ϕ_t: initial phase of transmitted (refracted) field (i.e., evaluated at $\underline{\mathbf{r}} = \underline{\mathbf{0}}, t = 0$)
- will show that:
  - $\omega_r = \omega_t = \omega_0$
  - $|\underline{\mathbf{k}}_r| = |\underline{\mathbf{k}}_0|$
  - and determine intrinsic phases ϕ_r and ϕ_t

1.2 Snell's Law for Reflection and Refraction of Waves

1.2.1 Boundary condition: Phases of three waves match at interface ($z = 0$) at all times t:

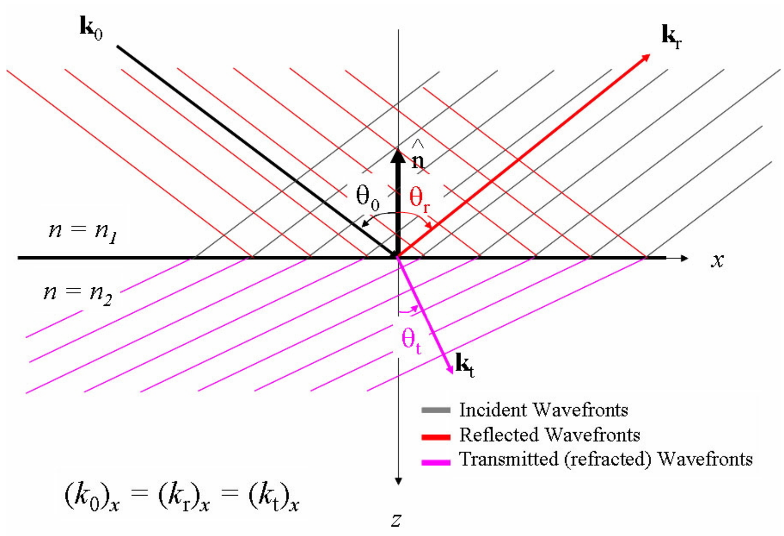
$$\begin{aligned}
\left.(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t)\right|_{z=0} &= \left.(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega_r t + \phi_r)\right|_{z=0} = \left.(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega_t t + \phi_t)\right|_{z=0} \\
\left.(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t)\right|_{z=0} &= (\underline{\mathbf{k}}_0)_x \cdot x + (\underline{\mathbf{k}}_0)_y \cdot y + (\underline{\mathbf{k}}_0)_z \cdot 0 - \omega_0 t \\
\left.(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega_r t + \phi_r)\right|_{z=0} &= (\underline{\mathbf{k}}_r)_x \, x + (\underline{\mathbf{k}}_r)_y \cdot y + (\underline{\mathbf{k}}_r)_z \cdot 0 - \omega_r t + \phi_r \\
\left.(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega_t t + \phi_t)\right|_{z=0} &= (\underline{\mathbf{k}}_t)_x \, x + (\underline{\mathbf{k}}_t)_y \cdot y + (\underline{\mathbf{k}}_t)_z \cdot 0 - \omega_r t + \phi_t
\end{aligned}$$

- Implies that temporal frequencies ω of three waves must be identical ($\omega_r = \omega_t = \omega_0$)
  - Otherwise: phases would differ at different times
  - Therefore: temporal frequency **is invariant** across boundary
    - Equivalent to saying that "color" of light does not change as light propagates into different medium
- Cancel temporal parts of phase:

$$\begin{aligned}
\left.(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t)\right|_{z=0} &= \left.(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega_0 t + \phi_r)\right|_{z=0} = \left.(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega_0 t + \phi_t)\right|_{z=0} \\
&\Longrightarrow \left.(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}})\right|_{z=0} = \left.(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} + \phi_r)\right|_{z=0} = \left.(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} + \phi_t)\right|_{z=0}
\end{aligned}$$

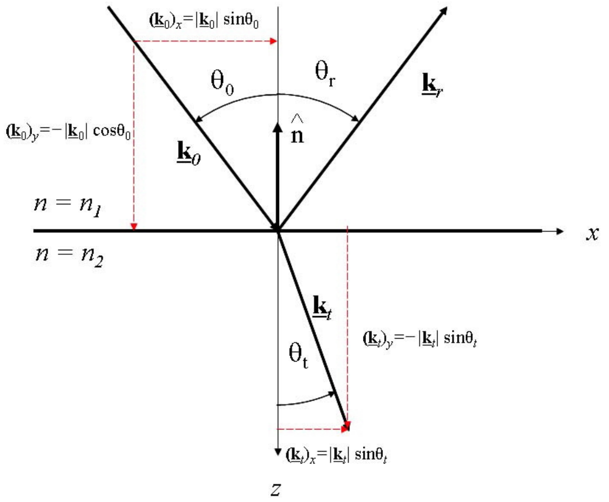
- Scalar products of three wavevectors with same position vector $\underline{\mathbf{r}}$ must be equal $\Longrightarrow$ three vectors $\underline{\mathbf{k}}_0$, $\underline{\mathbf{k}}_r$ and $\underline{\mathbf{k}}_t$ must lie in same plane
  - Assume $x - z$ plane
- Number of waves per unit length at any instant of time must match at boundary for all three waves
  - $\Longrightarrow$ x-components of three wavevectors must be equal:

$$(k_0)_x = (k_r)_x = (k_t)_x$$



The x-components of the three wavevectors (for the incident, reflected, and transmitted refracted waves) must be equal at the interface to ensure that each produces the same number of waves per unit length along the interface, so that the three wavefronts "match" despite the difference in wavelengths in the two media.

- From wavevectors as drawn:
  - reflected angle is *clockwise* from normal $\Longrightarrow$ negative angle



The $\underline{\mathbf{k}}$ vectors of the incident, reflected, and "transmitted" (refracted) wave at the interface between two media of index n_1 and n_2 (where $n_2 > n_1$ in the example shown).

$$\begin{aligned}
(k_0)_x &= |\underline{\mathbf{k}}_0| \sin[\theta_0] \\
(k_0)_z &= -\, |\underline{\mathbf{k}}_0| \cos[\theta_0] \\
(k_r)_x &= |\underline{\mathbf{k}}_r| \sin[\theta_r] = |\underline{\mathbf{k}}_0| \sin[\theta_r] \\
(k_r)_z &= +\, |\underline{\mathbf{k}}_r| \cos[\theta_r] = |\underline{\mathbf{k}}_0| \cos[\theta_r]
\end{aligned}$$

  - Lengths of incident and reflected wavevectors are equal because they are in the same medium:

$$\begin{aligned}
(k_0)_x = (k_r)_x &= |\underline{\mathbf{k}}_0| \sin[+\theta_0] = |\underline{\mathbf{k}}_r| \sin[+\theta_r] \\
&\Longrightarrow |\underline{\mathbf{k}}_0| \sin[\theta_0] = |\underline{\mathbf{k}}_0| \sin[-\,|\theta_r|] \\
&\Longrightarrow \sin[+\,|\theta_0|] = \sin[-\,|\theta_r|] \\
&\Longrightarrow \boxed{|\theta_r| = -\,|\theta_0|}
\end{aligned}$$

- Geometrical Law of Reflection

1.2.2 Transmission (= Refraction)

- Angle measured counter-clockwise from normal, $\theta_t > 0$:
- x-components of wavevectors must match at interface:

$$\begin{aligned}
(k_0)_x &= |\underline{\mathbf{k}}_0| \sin[\theta_0] = \frac{2\pi n_1}{\lambda_0} \sin[\theta_0] \\
(k_t)_x &= |\underline{\mathbf{k}}_t| \sin[\theta_t] = \frac{2\pi n_2}{\lambda_0} \sin[\theta_t]
\end{aligned}$$

- Relationship of angles of incident and transmitted wavevectors:

$$\begin{aligned}
&\frac{2\pi n_1}{\lambda_0} \sin[\theta_0] = \frac{2\pi n_2}{\lambda_0} \sin[\theta_t] \\
&\Longrightarrow \boxed{n_1 \sin[\theta_0] = n_2 \sin[\theta_t]}
\end{aligned}$$

- $\Longrightarrow$ Snell's law for refraction.

- Reflection law = Snell's refraction law if n is negative for reflected beam:

$$\begin{aligned}
n_1 \sin[\theta_0] &= -n_1 \sin[\theta_r] \\
&\Longrightarrow \sin[\theta_r] = -\sin[\theta_0] \\
&\Longrightarrow \theta_r = -\theta_0
\end{aligned}$$

1.3 Boundary Conditions for Electric and Magnetic Fields

- Snell's law: $\Longrightarrow$ angles of reflected and transmitted (refracted) plane waves
- "Quantity" of light reflected and refracted TBD
  - Geometries of fields depend on directions of electric field vectors
    - ∗ depends on "orientation" of electric field relative to interface $\Longrightarrow$ "polarization" of electric field
- Match appropriate boundary conditions at boundary
  - Apply to vector components of electric and magnetic fields on each side of boundary
  - Faraday's Law (MKS)

$$\boldsymbol{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}$$

  - Ampère's Law

$$\boldsymbol{\nabla} \times \underline{\mathbf{B}} = +\epsilon\mu \frac{\partial \underline{\mathbf{E}}}{\partial t} = +\frac{1}{\mathrm{v}_\phi^2}\frac{\partial \underline{\mathbf{E}}}{\partial t}$$

    - ∗ Phase Velocity

$$\mathrm{v}_\phi = \sqrt{\frac{1}{\epsilon\mu}}$$

  - assume incident field in form of plane wave:

$$\begin{aligned}
\underline{\mathbf{E}}_{\text{incident}}\,[x,y,z,t] &= \underline{\mathbf{E}}_0 \;\exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right] \\
&\equiv \underline{\mathbf{E}}_0 \;\exp\left[+i\left(\left[\underline{\mathbf{k}}_0\right]_x x + \left[\underline{\mathbf{k}}_0\right]_y y + \left[\underline{\mathbf{k}}_0\right]_z z - \omega_0 t\right)\right] \\
&\equiv \underline{\mathbf{E}}_0 \;\exp\left[+i\left(k_{0x}x + k_{0y}y + k_{0z}z - \omega_0 t\right)\right] \\
&= \left(\hat{\underline{\mathbf{x}}}E_{0x} + \hat{\underline{\mathbf{y}}}E_{0y} + \hat{\underline{\mathbf{z}}}E_{0z}\right)\exp\left[+i\left(k_{0x}x + k_{0y}y + k_{0z}z - \omega_0 t\right)\right]
\end{aligned}$$

- Assume direction of incident propagation lies in $x - z$ plane (defined by $\underline{\mathbf{k}}_0$ and $\hat{\underline{\mathbf{n}}}$) $\Longrightarrow$ $k_{0y} = 0$:

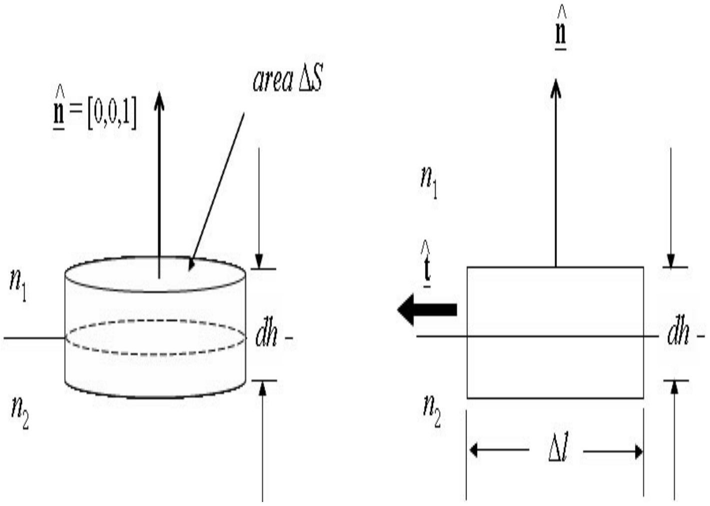
$$\begin{aligned}
\underline{\mathbf{E}}_{\text{incident}}\,[x,y,z,t] &= \underline{\mathbf{E}}_0 \;\exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right] \\
&= \left(\hat{\underline{\mathbf{x}}}E_{0x} + \hat{\underline{\mathbf{y}}}E_{0y} + \hat{\underline{\mathbf{z}}}E_{0z}\right)\exp\left[+i\left(k_{0x}x + k_{0z}z - \omega_0 t\right)\right]
\end{aligned}$$

- Now apply boundary conditions

1.4 Boundary Conditions satisfied by Fields at Boundary

1. Normal Components of $\underline{\mathbf{D}}$ and $\underline{\mathbf{B}}$ fields are continuous at boundary:

$$\begin{aligned} (\underline{\mathbf{D}}_1)_\perp &= (\underline{\mathbf{D}}_2)_\perp \implies (\underline{\mathbf{D}}_1)_z = (\underline{\mathbf{D}}_2)_z \\ (\underline{\mathbf{B}}_1)_\perp &= (\underline{\mathbf{B}}_2)_\perp \implies (\underline{\mathbf{B}}_1)_z = (\underline{\mathbf{B}}_2)_z \end{aligned}$$



The boundary conditions on the normal components of the electric and magnetic fields.

- Assume no charge or current on surface and within cylinder that straddles boundary
- If height dh of cylinder is decreased towards zero
  - $\implies$ Flux of electric and magnetic fields through top and bottom of cylinder (z-components, also called "normal components" in this geometry) must match
  - Gauss's laws:

$$\begin{gathered} \epsilon_1 \underline{\mathbf{E}}_1 \bullet \hat{\underline{\mathbf{n}}} - \epsilon_2 \underline{\mathbf{E}}_2 \bullet \hat{\underline{\mathbf{n}}} = 0 \\ \implies \epsilon_1 E_{1z} = \epsilon_2 E_{2z} \end{gathered}$$

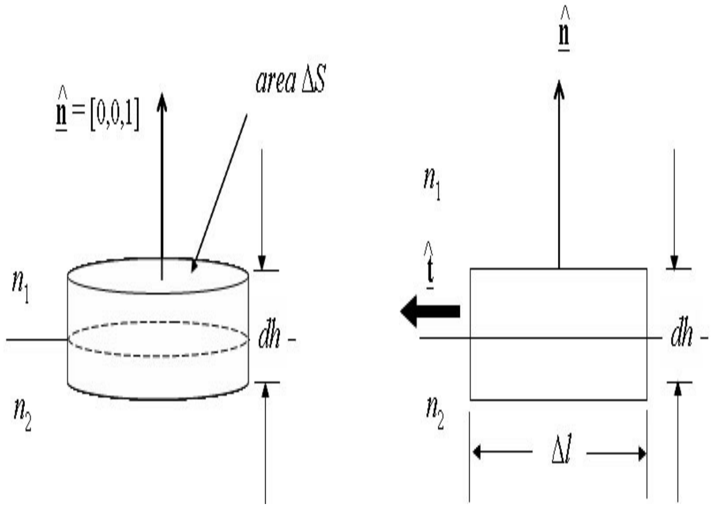
$$\begin{gathered} \underline{\mathbf{B}}_1 \bullet \hat{\underline{\mathbf{n}}} - \underline{\mathbf{B}}_2 \bullet \hat{\underline{\mathbf{n}}} = 0 \\ \implies B_{1z} = B_{2z} \end{gathered}$$

-
  -
    - flux of electric field in a medium is "displacement" field $\underline{\mathbf{D}} = \epsilon \underline{\mathbf{E}}$
    - flux of magnetic field is field $\underline{\mathbf{B}}$
  - Gauss' law $\implies$ "normal" components of $\underline{\mathbf{D}}$ and of $\underline{\mathbf{B}}$ *are continuous across boundary of medium*

2. Tangential (Parallel) Components of Fields $\underline{\mathbf{E}}$ and $\underline{\mathbf{H}} = \frac{\underline{\mathbf{B}}}{\mu}$

$$\begin{aligned}
(\underline{\mathbf{E}}_1)_{\parallel} &= (\underline{\mathbf{E}}_2)_{\parallel} \Longrightarrow (\underline{\mathbf{E}}_1)_x = (\underline{\mathbf{E}}_2)_x \\
\left(\frac{\underline{\mathbf{B}}_1}{\mu_1}\right)_{\parallel} &= \left(\frac{\underline{\mathbf{B}}_2}{\mu_2}\right)_{\parallel} \Longrightarrow (\underline{\mathbf{B}}_1)_x = (\underline{\mathbf{B}}_2)_x \ \text{(if } \mu_1 = \mu_2\text{)}
\end{aligned}$$

- Rectangular path ("loop") also straddles boundary



The tangential components of the electric and magnetic fields

- – unit vector $\hat{\underline{\mathbf{t}}} \perp \hat{\underline{\mathbf{n}}}$ points along interface surface. If "height" of loop $dh \to 0$, then circulations of electric and magnetic fields must cancel:

$$\underline{\mathbf{E}}_1 \bullet \hat{\underline{\mathbf{t}}} - \underline{\mathbf{E}}_2 \bullet \hat{\underline{\mathbf{t}}} = 0$$
$$\Longrightarrow E_{1x} = E_{2x}$$

$$\frac{\underline{\mathbf{B}}_1}{\mu_1} \bullet \hat{\underline{\mathbf{t}}} - \frac{\underline{\mathbf{B}}_2}{\mu_2} \bullet \hat{\underline{\mathbf{t}}} = 0$$
$$\Longrightarrow \frac{B_{1x}}{\mu_1} = \frac{B_{2x}}{\mu_2}$$

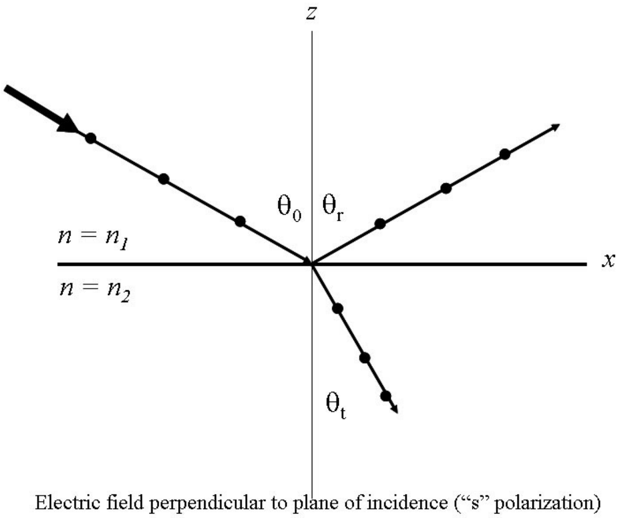
1.5 Use Boundary Conditions to:

- Solve Maxwell's equations for incident plane wave
  - result depends on incident angle θ_0 and on vector direction of electric field.

1.5.1 Two Cases:

1. Linear polarization perpendicular to plane of incidence defined by $\underline{\hat{\mathbf{n}}}$ and $\underline{\mathbf{k}}_0$

- "transverse electric" = "TE"
  - also called "*s polarization*"
  - or "*perpendicular*" or "$\perp$" polarization (used by Hecht)
  - Boundary conditions to be satisfied:
    1. tangential components of $\underline{\mathbf{E}}$ are continuous $\Longrightarrow$ $(\underline{\mathbf{E}}_0)_x = (\underline{\mathbf{E}}_r)_x$
    2. tangential components of $\underline{\mathbf{H}}$ are continuous $\Longrightarrow$ $\left(\frac{\underline{\mathbf{B}}_0}{\mu_1}\right)_x = \left(\frac{\underline{\mathbf{B}}_r}{\mu_1}\right)_x$

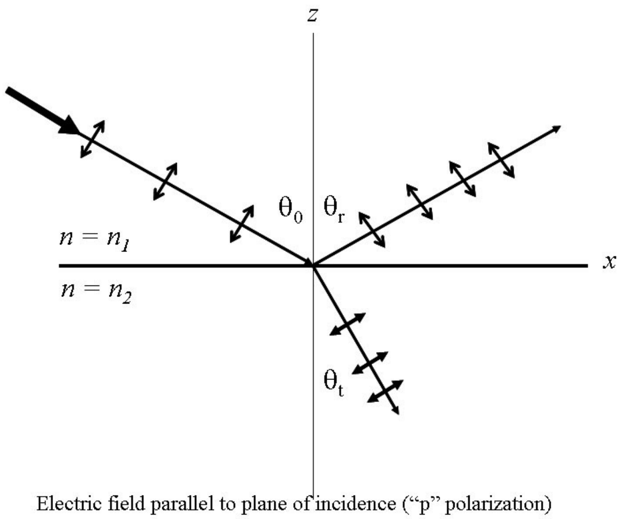


Electric field perpendicular to plane of incidence ("s" polarization)

The electric field perpendicular to the plane of incidence; this is the TRANSVERSE ELECTRIC field (TE, also called the "s" or "$\perp$" polarization).

2. Linear polarization in plane of incidence defined by $\underline{\hat{\mathbf{n}}}$ and $\underline{\mathbf{k}}_0$

- "transverse magnetic" = "TM" polarization (since magnetic field $\underline{\mathbf{B}}$ ⊥ plane of wavevectors $\underline{\mathbf{k}}_n$
  - "*p polarization*"
  - "*parallel*" or "||" polarization
  - Boundary conditions to be satisfied:Boundary conditions to be satisfied:
    1. normal components of $\underline{\mathbf{D}}$ are continuous $\Longrightarrow$ $\varepsilon_1\left(\underline{\mathbf{E}}_0\right)_z = \varepsilon_2\left(\underline{\mathbf{E}}_t\right)_z$
    2. tangential components of $\underline{\mathbf{E}}$ are continuous $\Longrightarrow$ $\left(\underline{\mathbf{E}}_0\right)_x = \left(\underline{\mathbf{E}}_r\right)_x$
    3. tangential components of $\underline{\mathbf{H}}$ are continuous $\Longrightarrow$ $\left(\frac{\underline{\mathbf{B}}_0}{\mu_1}\right)_x = \left(\frac{\underline{\mathbf{B}}_r}{\mu_1}\right)_x$



Electric field parallel to plane of incidence ("p" polarization)

The electric field is parallel to the plane of incidence; this is the TRANSVERSE MAGNETIC field (TM, also called the "p" or "||" polarization).

1.6 1. Transverse Electric (TE) Waves = “s” = “$\perp$” Polarization

- $\underline{\mathbf{E}}_0$ oriented along y direction (“into” or “out of” paper): $\underline{\mathbf{E}}_0 = \underline{\hat{\mathbf{y}}} \cdot |\underline{\mathbf{E}}_0|$
- $\underline{\mathbf{k}}_0$ has components in x and z directions: $\underline{\mathbf{k}}_0 = [k_{0x}, 0, k_{0z}] = \left[\frac{2\pi n_1}{\lambda_0}\sin[\theta_0], 0, -\frac{2\pi n_1}{\lambda_0}\cos[\theta_0]\right]$, $|\underline{\mathbf{k}}_0| = \sqrt{k_{0x}^2 + k_{0z}^2} = \frac{2\pi n_1}{\lambda_0}$

$$\begin{aligned}\underline{\mathbf{E}}_0[x,y,z,t] &= \left(\underline{\hat{\mathbf{x}}} \cdot 0 + \underline{\hat{\mathbf{y}}} \cdot |\underline{\mathbf{E}}_0| + \underline{\hat{\mathbf{z}}} \cdot 0\right) \exp\left[+i\left(k_{0x}x + k_{0z}z - \omega_0 t\right)\right] \\ &= \underline{\hat{\mathbf{y}}} E_0 \exp\left[+i\left(k_{0x}x + k_{0z}z - \omega_0 t\right)\right]\end{aligned}$$

- Magnetic field derived from:

$$\underline{\mathbf{E}} \times \underline{\mathbf{B}} \propto \underline{\mathbf{k}}_0$$

Since $\underline{\mathbf{k}}_0 \perp \underline{\mathbf{E}}$ and $\underline{\mathbf{k}}_0 \perp \underline{\mathbf{B}}$

$$\underline{\mathbf{B}} \propto \underline{\mathbf{k}}_0 \times \underline{\mathbf{E}} = \frac{n}{c}\frac{\underline{\mathbf{k}}_0 \times \underline{\mathbf{E}}}{|\underline{\mathbf{k}}_0|}$$

In fact:

$$\underline{\mathbf{B}} = \frac{n}{c}\frac{\underline{\mathbf{k}}_0 \times \underline{\mathbf{E}}}{|\underline{\mathbf{k}}_0|}$$

$$\begin{aligned}\underline{\mathbf{k}}_0 \times \underline{\mathbf{E}} &= \det\begin{bmatrix} \underline{\hat{\mathbf{x}}} & \underline{\hat{\mathbf{y}}} & \underline{\hat{\mathbf{z}}} \\ k_{0x} & 0 & k_{0z} \\ 0 & |\underline{\mathbf{E}}_0| & 0 \end{bmatrix} \\ &= \det\begin{bmatrix} \underline{\hat{\mathbf{x}}} & \underline{\hat{\mathbf{y}}} & \underline{\hat{\mathbf{z}}} \\ \frac{2\pi n_1}{\lambda_0}\sin[\theta_0] & 0 & -\frac{2\pi n_1}{\lambda_0}\cos[\theta_0] \\ 0 & |\underline{\mathbf{E}}_0| & 0 \end{bmatrix} \\ &= \underline{\hat{\mathbf{x}}}\left(-\frac{2\pi n_1}{\lambda_0}|\underline{\mathbf{E}}_0|\cos[\theta_0]\right) + \underline{\hat{\mathbf{z}}}\left(\frac{2\pi n_1}{\lambda_0}|\underline{\mathbf{E}}_0|\sin[\theta_0]\right) \\ &= \underline{\hat{\mathbf{x}}}\left(-|\underline{\mathbf{k}}_0|\,|\underline{\mathbf{E}}_0|\cos[\theta_0]\right) + \underline{\hat{\mathbf{z}}}\left(|\underline{\mathbf{k}}_0|\,|\underline{\mathbf{E}}_0|\sin[\theta_0]\right)\end{aligned}$$

$$\underline{\mathbf{B}}_{incident}[x,y,z,t] = \frac{n_1}{c}\frac{\underline{\mathbf{k}}_0 \times \underline{\mathbf{E}}}{|\underline{\mathbf{k}}_0|} = \underline{\hat{\mathbf{x}}}\left(-\frac{n_1}{c}|\underline{\mathbf{E}}_0|\cos[\theta_0]\right) + \underline{\hat{\mathbf{z}}}\left(\frac{n_1}{c}|\underline{\mathbf{E}}_0|\cos[\theta_0]\right)$$

$$= \left(\left[-\cos[\theta_0] \cdot n_1 \frac{|\underline{\mathbf{E}}_0|}{c}\right]\underline{\hat{\mathbf{x}}} + 0 \cdot \underline{\hat{\mathbf{y}}} + \left[+\sin[\theta_0] \cdot n_1 \frac{|\underline{\mathbf{E}}_0|}{c}\right]\underline{\hat{\mathbf{z}}}\right) \cdot \exp\left[+i\left(k_{0x}x + k_{0z}z - \omega_0 t\right)\right]$$

- Reflected fields:

$$\underline{\mathbf{E}}_r[x,y,z,t] = \underline{\hat{\mathbf{y}}} \cdot |\underline{\mathbf{E}}_r| \exp\left[+i\left(k_{rx}x + k_{rz}z - \omega_0 t\right)\right]$$

$$\underline{\mathbf{B}}_r\,[x,y,z,t] = \frac{n_1}{c}\frac{\underline{\mathbf{k}}_r \times \underline{\mathbf{E}}}{|\underline{\mathbf{k}}_r|}$$
$$= \left(\left[+\cos\left[-\theta_0\right]\cdot n_1\frac{|\underline{\mathbf{E}}_r|}{c}\right]\hat{\underline{\mathbf{x}}} + \left[-\sin\left[-\theta_0\right]\cdot n_1\frac{|\underline{\mathbf{E}}_r|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\cdot\exp\left[+i\left(k_{rx}x + k_{rz}z - \omega_0 t\right)\right]$$
$$= \left(\left[+\cos\left[\theta_0\right]\cdot n_1\frac{|\underline{\mathbf{E}}_r|}{c}\right]\hat{\underline{\mathbf{x}}} + \left[\sin\left[\theta_0\right]\cdot n_1\frac{|\underline{\mathbf{E}}_r|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\cdot\exp\left[+i\left(k_{rx}x + k_{rz}z - \omega_0 t\right)\right]$$

- Transmitted (refracted) fields:

$$\underline{\mathbf{E}}_t\,[x,y,z,t] = \hat{\underline{\mathbf{y}}}\cdot|\underline{\mathbf{E}}_t|\exp\left[+i\left(k_{tx}x + k_{tz}z - \omega_0 t\right)\right]$$

$$\underline{\mathbf{B}}_t\,[x,y,z,t] = \frac{n_2}{c}\frac{\underline{\mathbf{k}}_t \times \underline{\mathbf{E}}_t}{|\underline{\mathbf{k}}_t|}$$
$$= \left(\left[-\cos\left[\theta_t\right]\cdot n_2\frac{|\underline{\mathbf{E}}_t|}{c}\right]\hat{\underline{\mathbf{x}}} + \left[\sin\left[\theta_t\right]\cdot n_2\frac{|\underline{\mathbf{E}}_t|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\cdot\exp\left[+i\left(k_{tx}x + k_{tz}z - \omega_0 t\right)\right]$$

- Electric field at interface has ONLY transverse (tangential) components
  - Boundary condition $\Longrightarrow$ tangential electric field is continuous:

$$E_0 + E_r = E_t \Longrightarrow 1 + \frac{E_r}{E_0} = \frac{E_t}{E_0}$$

- Reflection and Transmission Coefficients for amplitude:

$$r_{TE} \equiv \frac{E_r}{E_0}$$
$$t_{TE} \equiv \frac{E_t}{E_0}$$

- Boundary condition for normal magnetic field $\Longrightarrow$

$$\frac{n_1}{c}\sin\left[\theta_0\right]\left(E_0 + E_r\right) = \frac{n_2}{c}\sin\left[\theta_t\right]E_t$$

- Boundary condition for tangential magnetic field $\Longrightarrow$

$$\frac{n_1}{\mu_1 c}\cos\left[\theta_0\right]\left(E_0 - E_r\right) = \frac{n_2}{\mu_2 c}\cos\left[\theta_t\right]E_t$$

  - Solve simultaneously for r and t to find Reflectance Coefficient

Reflectance Coefficient for TE Waves

$$r_{TE} = \frac{E_r}{E_0} = \frac{\frac{n_1}{\mu_1}\cos\left[\theta_0\right] - \frac{n_2}{\mu_2}\cos\left[\theta_t\right]}{\frac{n_1}{\mu_1}\cos\left[\theta_0\right] + \frac{n_2}{\mu_2}\cos\left[\theta_t\right]}$$

$$\boxed{r_{TE} = \frac{n_1\cos\left[\theta_0\right] - n_2\cos\left[\theta_t\right]}{n_1\cos\left[\theta_0\right] + n_2\cos\left[\theta_t\right]}}\ \textit{if}\ \mu_1 = \mu_2\ \text{(usual case)}$$

- This is the first *Fresnel Equation*
    - Normal incidence $\Longrightarrow$ $\theta_0 = 0$ $\Longrightarrow$ $\theta_t = 0$ from Snell's Law:

$$r_{TE}\left[\theta_0 = 0\right] = \frac{n_1 - n_2}{n_1 + n_2}$$

    - $n_1 = 1.0$ (air) and $n_2 = 1.5$ (glass) $\Longrightarrow$ $r_{TE}\left[\theta_0 = 0\right] = \dfrac{1 - 1.5}{1 + 1.5} = -0.2$

- Amplitude Transmittance Coefficient t_{TE}:

Transmission Coefficient for TE Waves

$$\begin{aligned} t_{TE} &= \frac{E_t}{E_0} = 1 + \frac{E_r}{E_0} \\ &= 1 + \frac{n_1 \cos\left[\theta_0\right] - n_2 \cos\left[\theta_t\right]}{n_1 \cos\left[\theta_0\right] + n_2 \cos\left[\theta_t\right]} \\ &= \frac{n_1 \cos\left[\theta_0\right] + n_2 \cos\left[\theta_t\right] + \left(n_1 \cos\left[\theta_0\right] - n_2 \cos\left[\theta_t\right]\right)}{n_1 \cos\left[\theta_0\right] + n_2 \cos\left[\theta_t\right]} \end{aligned}$$

$$\Longrightarrow \boxed{t_{TE} = \frac{+2 n_1 \cos\left[\theta_0\right]}{n_1 \cos\left[\theta_0\right] + n_2 \cos\left[\theta_t\right]}} \; \textit{if } \mu_1 = \mu_2$$

  - Normal incidence $\theta_0 = 0 \Longrightarrow \theta_t = 0$

$$t_{TE} = \frac{+2 n_1}{n_1 + n_2}$$

  - $n_1 = 1.0$ (air) and $n_2 = 1.5$ (glass) $\Longrightarrow$ $t_{TE} = \frac{2}{2.5} = 0.8$

- **These are** *Amplitude Coefficients*

- *Reflectance* and *Transmittance* measure ratios of reflected or transmitted power to incident power.

- Reflectance
  - Power proportional to product of magnitude of Poynting vector and area of beam.
  - Areas of beams before and after *reflection* are identical
  - Reflectance is just ratio of magnitudes of Poynting vectors:

$$\boxed{R_{TE} = r_{TE}^2}$$

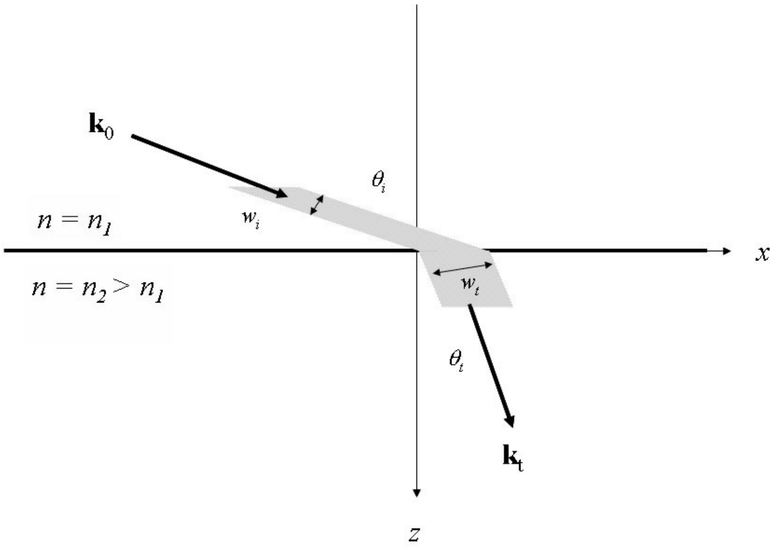
$$\boxed{R_{TE} = \left(\frac{n_1 \cos\left[\theta_0\right] - n_2 \cos\left[\theta_t\right]}{n_1 \cos\left[\theta_0\right] + n_2 \cos\left[\theta_t\right]}\right)^2}$$

- If $\theta_0 = 0 \implies \theta_t = 0$ (normal incidence):

$$R_{TE} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

- $n_1 = 1.0$ (air) and $n_2 = 1.5$ (glass) $\implies R_{TE} = \left(\frac{1-1.5}{1+1.5}\right)^2 = 0.04$

- Transmittance T_{TE}
  - more complicated to compute because width of beam changes in one direction (and thus area)
  - Example with $n_1 > n_2$
    - Wider transmitted ("refracted") beam along x-axis
    - Area of transmitted beam is *larger* in medium with *larger* index:



Demonstration that the cross-sectional area of the beam is changed by refraction at the interface between two media with different refractive indices. The area is larger in the medium with the larger index. This difference must be accounted for in the calculation of the power transmission T across the interface.

  - Magnitude of Poynting vector proportional to product of index of refraction and squared magnitude of electric field:

$$|\underline{\mathbf{s}}_1| \propto n_1 |E_0|^2$$
$$|\underline{\mathbf{s}}_2| \propto n_2 |E_t|^2$$

- Ratio of transmitted to incident power:

$$T = \frac{|\underline{\mathbf{s}}_2| \cdot A_2}{|\underline{\mathbf{s}}_1| \cdot A_1} = \frac{\left(n_2 \left|E_t\right|^2\right) \cdot A_2}{\left(n_1 \left|E_0\right|^2\right) \cdot A_1} = \frac{n_2}{n_1} \cdot t^2 \cdot \frac{A_2}{A_1}$$

- Area of transmitted beam changes in proportion to dimension along x-axis:

$$\begin{aligned} \frac{A_2}{A_1} &= \frac{w_2}{w_1} = \frac{\sin\left[\frac{\pi}{2} - \theta_t\right]}{\sin\left[\frac{\pi}{2} - \theta_0\right]} = \frac{\cos[\theta_t]}{\cos[\theta_0]} \\ &> 1 \ \textit{if} \ \ |\theta_t| < |\theta_0| \ \ \textit{or} \ n_2 > n_1 \end{aligned}$$

$$T = \frac{n_2}{n_1} \cdot t^2 \cdot \left(\frac{\cos[\theta_t]}{\cos[\theta_0]}\right)$$

$$\boxed{T = \left(\frac{n_2 \cos[\theta_t]}{n_1 \cos[\theta_0]}\right) \cdot t^2}$$

  - normal incidence $\Longrightarrow$ cross-sectional area is constant

$$T = \left(\frac{n_2 \cos[0]}{n_1 \cos[0]}\right) \cdot t^2$$

  - $n_1 = 1$, $n_2 = 1.5 \Longrightarrow$

$$T = \frac{n_2}{n_1} \cdot \left(\frac{+2n_1}{n_1 + n_2}\right)^2 = \frac{1.5}{1} \cdot \left(\frac{2}{2.5}\right)^2 = 0.96 = 1 - R$$

- Add Snell's law to rewrite equation:

$$n_1 \sin[\theta_0] = n_2 \sin[\theta_t]$$

$$\Longrightarrow \sin[\theta_t] = \frac{n_1}{n_2} \sin[\theta_0]$$

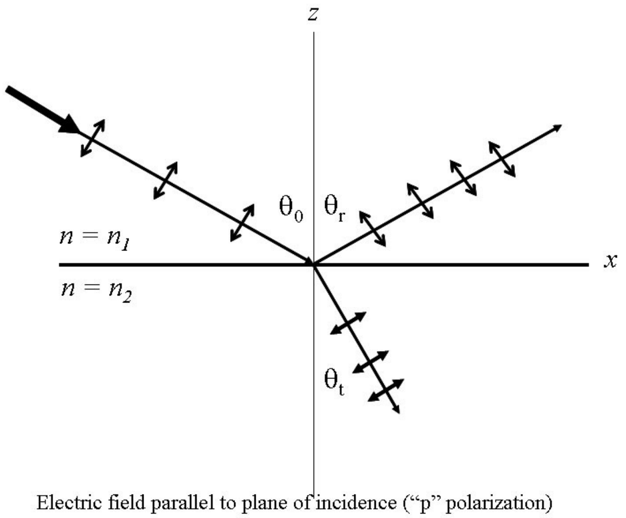
$$\Longrightarrow \cos[\theta_t] = \sqrt{1 - \sin^2[\theta_t]} = \sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2} = \frac{1}{n_2}\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}$$

- $\Longrightarrow$

$$T = \frac{n_2 \cos[\theta_t]}{n_1 \cos[\theta_0]} \cdot t^2 = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}}{n_1 \cos[\theta_0]}\right) \cdot t^2$$

$$\boxed{T_{TE} = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}}{n_1 \cos[\theta_0]}\right) \cdot \left(\frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}\right)^2}$$

1.7 2. Transverse Magnetic (TM) Waves = “p” = “$\|$” polarization



Electric field parallel to plane of incidence (“p” polarization)

The electric field is parallel to the plane of incidence; this is the TRANSVERSE MAGNETIC field (TM, also called the “p” or “$\|$” polarization).

- Electric field is in x-z plane
- Wavevector has components in x and z directions:

$$\begin{aligned}\underline{\mathbf{E}}_{incident}[x,y,z,t] &= \left(\underline{\hat{\mathbf{x}}} \cdot |\underline{\mathbf{E}}_0| \cos[\theta_0] + \underline{\hat{\mathbf{y}}} \cdot 0 + \underline{\hat{\mathbf{z}}} \cdot |\underline{\mathbf{E}}_0| \sin[-\theta_0]\right) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)] \\ &= (\underline{\hat{\mathbf{x}}} \cdot |\underline{\mathbf{E}}_0| \cos[\theta_0] - \underline{\hat{\mathbf{z}}} \cdot |\underline{\mathbf{E}}_0| \sin[\theta_0]) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)]\end{aligned}$$

- Magnetic field in y-direction:

$$\underline{\mathbf{B}}_{incident}[x,y,z,t] = \left(n_1 \frac{|\underline{\mathbf{E}}_0|}{c} \underline{\hat{\mathbf{y}}}\right) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)]$$

- Reflected fields:

$$\begin{aligned}\underline{\mathbf{E}}_{reflected}[x,y,z,t] &= (\underline{\hat{\mathbf{x}}} \cdot - |\underline{\mathbf{E}}_0| \cos[\theta_0] - \underline{\hat{\mathbf{z}}} \cdot |\underline{\mathbf{E}}_0| \sin[\theta_0]) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)] \\ \underline{\mathbf{B}}_{reflected}[x,y,z,t] &= \left(n_1 \frac{|\underline{\mathbf{E}}_r|}{c} \underline{\hat{\mathbf{y}}}\right) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)]\end{aligned}$$

- Transmitted (refracted) fields:

$$\begin{aligned}\underline{\mathbf{E}}_{transmitted}[x,y,z,t] &= (\underline{\hat{\mathbf{x}}} \cdot |\underline{\mathbf{E}}_0| \cos[\theta t] - \underline{\hat{\mathbf{z}}} \cdot |\underline{\mathbf{E}}_0| \sin[\theta_t]) \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)] \\ \underline{\mathbf{B}}_{transmitted}[x,y,z,t] &= \left(n_2 \frac{|\underline{\mathbf{E}}_t|}{c} \underline{\hat{\mathbf{y}}}\right) \exp[+i(k_{tx}x + k_{tz}z - \omega_0 t)]\end{aligned}$$

- Boundary condition on normal component of $\underline{\mathbf{B}}$ is trivial
- Boundary condition on other components:

$$\begin{gathered}\varepsilon_1 \sin[\theta_0](E_0+E_r)=\varepsilon_2 \sin[\theta_t] E_t \\ \cos[\theta_0](E_0-E_r)=\cos[\theta_t] E_t \\ \frac{n_1}{\mu_1 c}(E_0+E_r)=\frac{n_2}{\mu_1 c} E_t\end{gathered}$$

- Reflection coefficients:

$$\begin{gathered}\textit{Transverse Magnetic Waves} \\ r_{TM}=\frac{+\frac{n_2}{\mu_2}\cos[\theta_0]-\frac{n_1}{\mu_1}\cos[\theta_t]}{+\frac{n_2}{\mu_2}\cos[\theta_0]+\frac{n_1}{\mu_1}\cos[\theta_t]}\end{gathered}$$

$$\boxed{r_{TM}=\frac{+n_2\cos[\theta_0]-n_1\cos[\theta_t]}{+n_2\cos[\theta_0]+n_1\cos[\theta_t]}}\ \textit{if}\ \mu_1=\mu_2$$

- Reflectance:

$$\boxed{R_{TM}=\left(\frac{+n_2\cos[\theta_0]-n_1\cos[\theta_t]}{+n_2\cos[\theta_0]+n_1\cos[\theta_t]}\right)^2}$$

- Transmission coefficient:

$$t_{TM}=\frac{2\frac{n_1}{\mu_1}\cos[\theta_0]}{+\frac{n_2}{\mu_2}\cos[\theta_0]+\frac{n_1}{\mu_1}\cos[\theta_t]}$$

$$\boxed{t_{TM}=\frac{2n_1\cos[\theta_0]}{+n_2\cos[\theta_0]+n_1\cos[\theta_t]}}\ \textit{if}\ \mu_1=\mu_2$$

- Transmittance:

$$\boxed{T_{TM}=\left(\frac{\sqrt{n_2^2-n_1^2\sin^2[\theta_0]}}{n_1\cos[\theta_0]}\right)\cdot\left(\frac{2n_1\cos[\theta_0]}{+n_2\cos[\theta_0]+n_1\cos[\theta_t]}\right)^2}$$

1.8 Comparison of TE and TM Coefficients

$$r_{TE} = \frac{n_1 \cos[\theta_0] - n_2 \cos[\theta_t]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}$$
$$r_{TM} = \frac{+n_2 \cos[\theta_0] - n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}$$

$$t_{TE} = \frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]} = \frac{+2\cos[\theta_0]}{\cos[\theta_0] + \frac{n_2}{n_1}\cos[\theta_t]}$$
$$t_{TM} = \frac{+2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]} = \frac{+2\cos[\theta_0]}{+\frac{n_2}{n_1}\cos[\theta_0] + \cos[\theta_t]}$$

- Snell's law:

$$n_1 \sin[\theta_0] = n_2 \sin[\theta_t]$$
$$\Longrightarrow \cos[\theta_t] = \sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}$$

$$r_{TE} = \frac{n_1 \cos[\theta_0] - n_2\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}{n_1 \cos[\theta_0] + n_2\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}$$
$$r_{TM} = \frac{+n_2 \cos[\theta_0] - n_1\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}{+n_2 \cos[\theta_0] + n_1\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}$$

$$t_{TE} = \frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}$$
$$t_{TM} = \frac{+2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1\sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}}$$

- TE Case $\Longrightarrow$ angles and indices from "same" media are combined (n_1 multiplies cosine of θ_0, which is in same medium)
- TM Case $\Longrightarrow$ angles and indices from "other" media are combined (n_1 is applied to $\cos[\theta_t]$ and n_2 to $\cos[\theta_0]$) except in numerator of t

1.9 Normal Incidence $(\theta_0 = \theta_r = \theta_t = 0)$

$$\begin{aligned}
r_{TE}|_{\theta_0=0} &= \frac{n_1 - n_2}{n_1 + n_2} \\
r_{TM}|_{\theta_0=0} &= \frac{+n_2 - n_1}{+n_2 + n_1} = -\left(r_{TE}|_{\theta_0=0}\right) \\
t_{TE}|_{\theta_0=0} &= \frac{+2n_1}{n_1 + n_2} \\
t_{TM}|_{\theta_0=0} &= \frac{+2n_1}{n_1 + n_2} = t_{TE}|_{\theta_0=0}
\end{aligned}$$

- Reflectance coefficients are different even though two polarizations are indistinguishable at normal incidence
- Areas of incident and transmitted waves are identical $\Longrightarrow$ there is no area factor in amplitude transmittance. resulting formulas for observable reflectance and transmittance are identical:

$$\textit{normal incidence} \;\; (\theta_0 = 0)$$

$$R_{TE}(\theta_0 = 0) = R_{TM}(\theta_0 = 0) \equiv \boxed{R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

$$\boxed{T = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

1.10 “Rare-to-Dense” Reflection at Normal Incidence

e.g., “air to glass” with $n_1 = 1.0 < n_2 = 1.5$

- Fresnel coefficients:

$$\begin{aligned} r_{TE} &= \frac{1.0 - 1.5}{1.0 + 1.5} = -0.2 = 0.2e^{+i\pi} \\ r_{TM} &= \frac{1.5 - 1.0}{1.5 + 1.0} = +0.2 \\ t_{TE} = t_{TM} &= \frac{2 \cdot 1.0}{1.0 + 1.5} = +0.8 \end{aligned}$$

for “rare-to-dense” reflection with $n_1 = 1.0$ *and* $n_2 = 1.5$

$r_{TE} < 0 \implies$ phase of reflected light changed by π radians $= 180^\circ$

- Reflectivity:

$$R_{TE} = R_{TM} = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = 0.04$$

- Transmittance

$$T_{TE} = T_{TM} = \frac{4 \cdot 1 \cdot 1.5}{(1 + 1.5)^2} = 0.96$$

- $\implies$

$$R + T = 1$$

1.11 “Dense-to-Rare” Reflection at Normal Incidence

$n_1 = 1.5 > n_2 = 1.0$

- Amplitude reflection coefficients at normal incidence:

$$\begin{aligned} r_{TE} &= \frac{1.5-1.0}{1.5+1.0} = +0.2 \\ r_{TM} &= \frac{1.0-1.5}{1.0+1.5} = -0.2 = 0.2e^{+i\pi} \end{aligned}$$

- Reflectivity same as “rare-to-dense” reflection:

$$R = (\pm 0.2)^2 = 0.04$$

- Amplitude transmission coefficients at normal incidence in a “dense-to-rare” reflection are identical:

$$t_{TE} = t_{TM} = \frac{2 \cdot 1.5}{1.0+1.5} = +1.2 > 1.0$$

- $t_{TM} = t_{TE} > 1$?
  - Transmittance requires additional geometrical factor:

$$\begin{aligned} T &= \left(\frac{n_2 \cos[\theta_t]}{n_1 \cos[\theta_0]}\right) \cdot t^2 \\ &= \left(\frac{1}{1.5}\right) \cdot (+1.2)^2 = 0.96 \end{aligned}$$

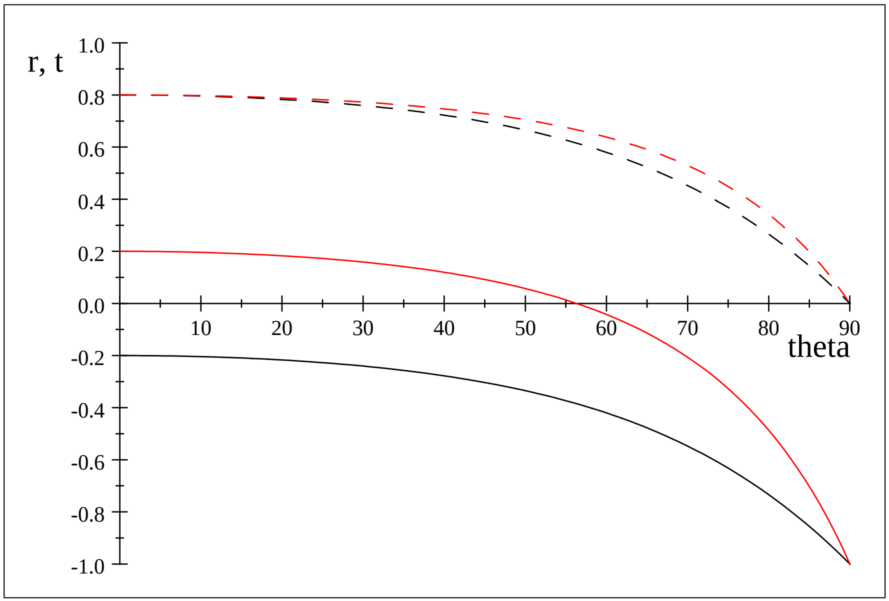
- Energy is conserved:

$$R + T = 1$$

1.12 Angular Dependence of Amplitude Coefficients at "Rare-to-Dense" Interface

$n_1 = 1 < n_2 = 1.5$

- Coefficients plotted vs. incident angle measured in degrees from 0° (normal incidence) to 90° (grazing incidence)
- $r_{TM} = 0$ at specific angle $\theta_B \cong 60°$ (Brewster's Angle):
  - $R_{TM} = 0$ and $T_{TM} = 1$ at $\theta_B \cong 60°$
  - Discovered in 1815 by David Brewster.
  -

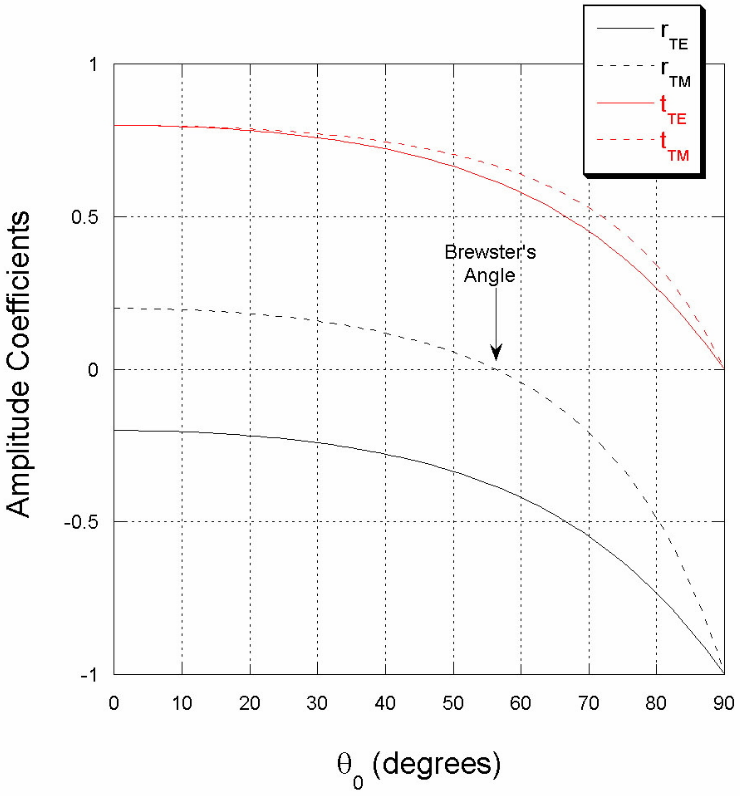


Amplitude coefficients r_{TE} (solid black), r_{TM} (dashed black), t_{TE} (solid red), and t_{TM} (dashed red); note that $r_{TM} = 0$ at Brewster's Angle and that $t_{TE} = t_{TM} = 0$ at $\theta_0 = \frac{\pi}{2}$ radians.

-

$$t_{TE} = \frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]} = \frac{+2\cos[\theta_0]}{\cos[\theta_0] + \frac{n_2}{n_1}\cos[\theta_t]}$$

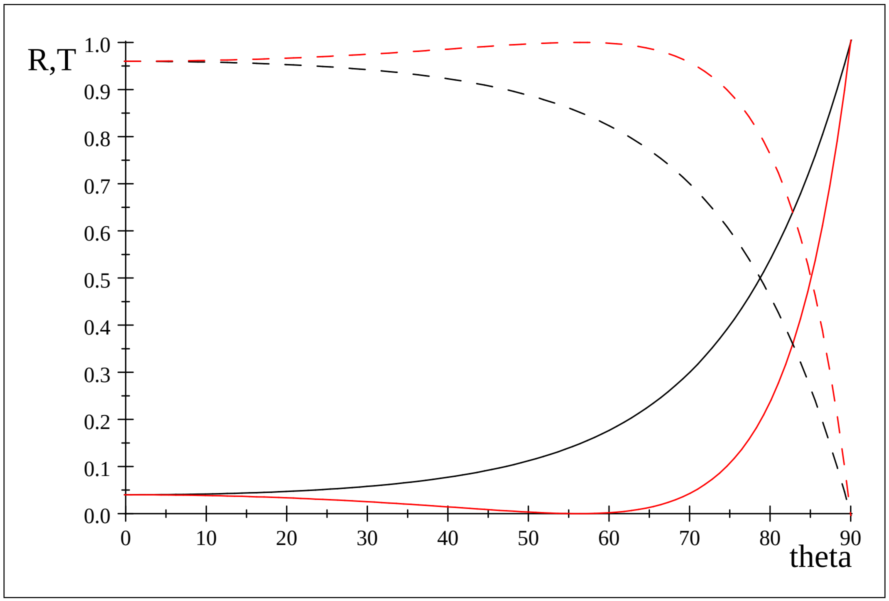
$$t_{TM} = \frac{+2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]} = \frac{+2\cos[\theta_0]}{+\frac{n_2}{n_1}\cos[\theta_0] + \cos[\theta_t]}$$



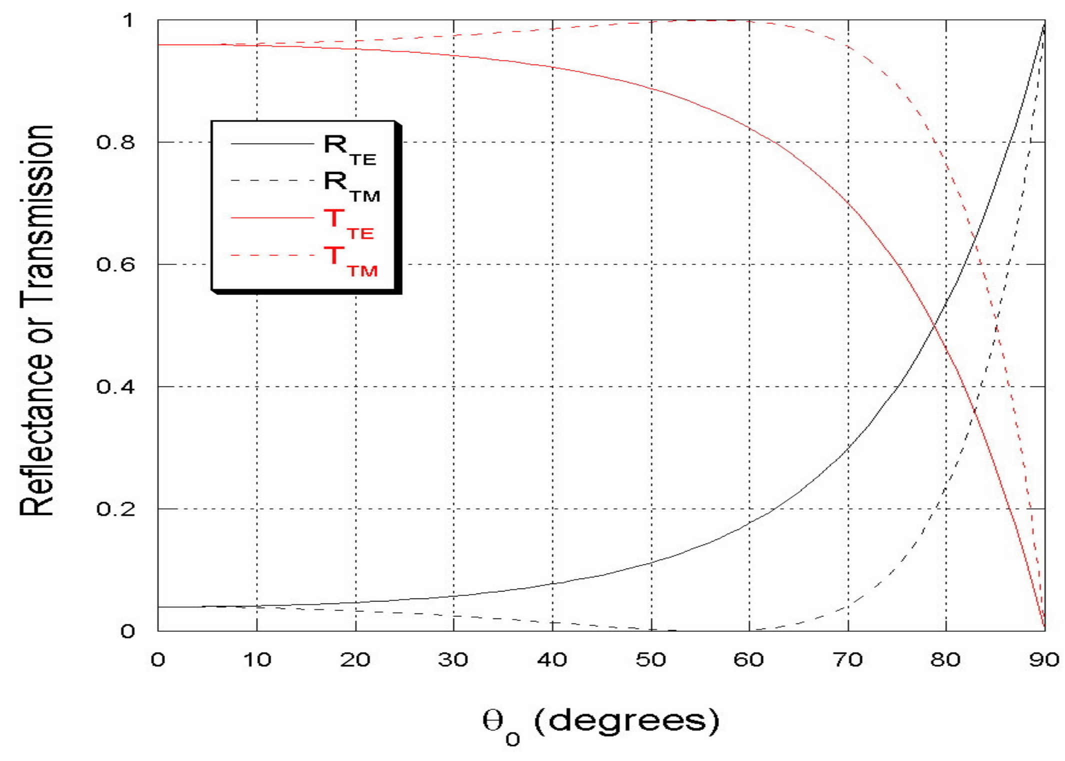
Amplitude reflectance and transmittance coefficients for "rare-to-dense" reflection: $n_1 = 1.0$ *(air) and* $n_2 = 1.5$ *(glass) for both TE and TM waves, plotted as functions of the incident angle from* $\theta_0 = 0°$ *(normal incidence) to* $\theta_0 = 90°$ *(grazing incidence). The reflectance coefficient* $r_{TE} < 0$ *for all* θ*, which means that there is a phase shift upon reflection, whereas* $r_{TM} > 0$ *for* $\theta_0 < \theta_B$ *(Brewster's angle). Also note that the transmittance coefficients are very similar functions.*

1.13 Reflectance and Transmittance at “Rare-to-Dense” Interface

$n_1 = 1.0$ and $n_2 = 1.5$



Reflectances R_{TE} (solid black), R_{TM} (dashed black), and transmittances T_{TE} (solid red), and T_{TM} (dashed red)



Reflectance and transmittance for $n_1 = 1.0$ and $n_2 = 1.5$ for TE and TM waves. Note that $R_{TM} = 0$ and $T_{TM} = 1$ at "Brewster's angle."

1.14 Complete Polarization of Reflected Wave – Brewster's Angle

- "Rare-to-Dense" reflection at Brewster's Angle $\theta_0 = \theta_B$
- $r_{TM} = 0 \implies R_{TM} = 0$
- Reflected light at TE polarization only
- Electrons driven in plane of incidence will not emit radiation at angle required by law of reflection
- *Angle of complete polarization*
- θ_B evaluated by setting $r_{TM} = 0$:

$$\begin{aligned} r_{TM} &= \frac{+n_2\cos[\theta_B] - n_1\cos[\theta_t]}{+n_2\cos[\theta_B] + n_1\cos[\theta_t]} = 0 \\ &\implies +n_2\cos[\theta_B] = n_1\cos[\theta_t] \\ &\implies \cos^2[\theta_t] = \left(\frac{n_2}{n_1}\right)^2 \cos^2[\theta_B] \end{aligned}$$

  - Expression for $\sin^2[\theta_t]$ from Snell's law:

$$n_1\sin[\theta_B] = n_2\sin[\theta_t] \implies \sin^2[\theta_t] = \left(\frac{n_1}{n_2}\right)^2 \sin^2[\theta_B]$$

  - Square and add to expression for $\cos^2[\theta_t]$:

$$\cos^2[\theta_t] + \sin^2[\theta_t] = 1 = \left(\frac{n_2}{n_1}\right)^2 \cos^2[\theta_B] + \left(\frac{n_1}{n_2}\right)^2 \sin^2[\theta_B]$$

  - $\cos^2[\theta_B] + \sin^2[\theta_B] = 1 \implies$

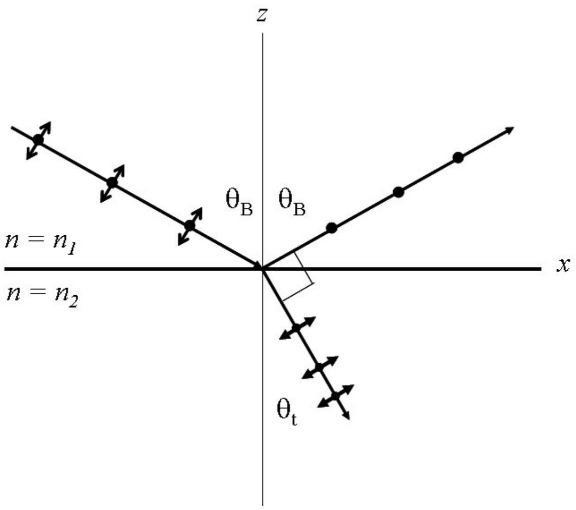
$$\begin{aligned} \cos^2[\theta_B] + \sin^2[\theta_B] &= \left(\frac{n_2}{n_1}\right)^2 \cos^2[\theta_B] + \left(\frac{n_1}{n_2}\right)^2 \sin^2[\theta_B] \\ &\implies \frac{n_2^2 - n_1^2}{n_1^2}\cos^2[\theta_B] + \frac{n_1^2 - n_2^2}{n_2^2}\sin^2[\theta_B] = 0 \\ &\implies \frac{\sin^2[\theta_B]}{\cos^2[\theta_B]} = \tan^2[\theta_B] = \frac{n_2^2}{n_1^2} \end{aligned}$$

$\implies$

$$\boxed{\theta_B = \tan^{-1}\left[\frac{n_2}{n_1}\right]}$$

  - $n_1 = 1$ (air) and $n_2 = 1.5$ (glass) $\implies \theta_B \cong 56.3^\circ$

- $\theta_0 > 56^\circ \implies$ reflected light is *plane polarized parallel to plane of incidence*



Polarization of reflected light at Brewster's angle. The incident beam at $\theta_0 = \theta_B$ is unpolarized. The reflectance coefficient for light polarized in the plane (TM waves) is 0, and the sum of the incident and refracted angle is $90^\circ = \frac{\pi}{2}$. Thus

$$\theta_B + \theta_t = \frac{\pi}{2} \Longrightarrow \theta_t = \frac{\pi}{2} - \theta_B.$$

- Find θ_t from Snell's law:

$$n_1 \sin[\theta_B] = n_2 \sin[\theta_t]$$

$$1 \cdot \sin\left[\tan^{-1}\left[\frac{1.5}{1}\right]\right] = 1.5 \cdot \sin[\theta_t]$$

$$\theta_t = \sin^{-1}\left[\frac{1}{1.5}\sin\left[\tan^{-1}\left[\frac{1.5}{1}\right]\right]\right] \cong 33.7^\circ$$

$$\theta_B + \theta_t = 56.3^\circ + 33.7^\circ = 90^\circ = \frac{\pi}{2} \text{ radians}$$

- Another expression for Brewster's Angle

$$\boxed{\theta_B + \theta_t = \tfrac{\pi}{2} \ \textit{radians}}$$

- Handy means to determine polarization axis of a linear polarizer
  - Look through linear polarizer at light reflected at shallow angle relative to surface (e.g., a waxed floor)
  - Transmitted (refracted) light contains both polarizations, though not in equal amounts

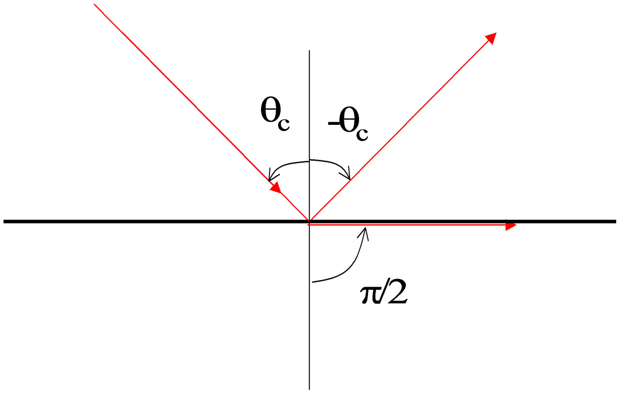
1.15 r and t at "Dense-to-Rare" Interface – Critical Angle

- "Dense-to-rare" interface $n_1 > n_2$
- $$t_{TE} = t_{TM} = 0 \text{ if } \theta_t = \frac{\pi}{2}$$

  Snell's Law:

  $$n_1 \sin\theta_C = n_2 \sin\left[\frac{\pi}{2}\right] = n_2$$

  $$\boxed{\theta_C = \sin^{-1}\left[\frac{n_2}{n_1}\right]}$$



Reflection at a "dense-to-rare" interface for light incident at the critical angle θ_c.

- If incident angle $\theta_0 > \theta_C$ (*critical angle*), $r_{TE} = r_{TM} = 1 \implies R_{TE} = R_{TM} = 1$
- Light incident at $\theta_0 > \theta_C$ is *totally reflected* at a "dense-to-rare" interface
- *Total Internal Reflectance* (TIR) *for incident angles* $\theta_0 > \theta_C = \sin^{-1}\left[\frac{n_2}{n_1}\right]$
  - "internal" $\implies$ within dense medium
  - optical fibers in communications.
  - $n_1 = 1.5$, $n_2 = 1.0 \implies$

    $$\sin[\theta_C] = \frac{2}{3} \implies \theta_0 \cong 0.73 \text{ radians } \cong 41.8^\circ \equiv \theta_C$$

  - Brewster's angle in same case:

    $$\theta_B = \tan^{-1}\left[\frac{n_2}{n_1}\right] = \tan^{-1}\left[\frac{2}{3}\right] \implies \theta_B \cong 0.59 \text{ radians} \cong 33.7^\circ < \theta_C$$

    Angular Dependence of Amplitude Coefficients at "Dense-to-Rare" Interface

$n_1 = 1.5 > n_2 = 1$

- $$\begin{aligned} r_{TE} &= \frac{+1\cos[\theta_0] - 1.5\cdot\sqrt{1-\sin^2[\theta_0]}}{+1.5\cos[\theta_0] + 1\cdot\sqrt{1-\sin^2[\theta_0]}} \\ r_{TM} &= \frac{+1\cos[\theta_0] - 1.5\cdot\sqrt{1-\sin^2[\theta_0]}}{+1\cos[\theta_0] + 1.5\cdot\sqrt{1-\sin^2[\theta_0]}} \end{aligned}$$

- Reflectance:
$$R_{TM} = \left(\frac{+n_2\cos[\theta_0] - n_1\cos[\theta_t]}{+n_2\cos[\theta_0] + n_1\cos[\theta_t]}\right)^2$$

–

1.16 Reflectance and Transmittance at “Dense-to-Rare” Interface

$n_1 = 1.5$ and $n_2 = 1.0$

Brewster’s Angle:

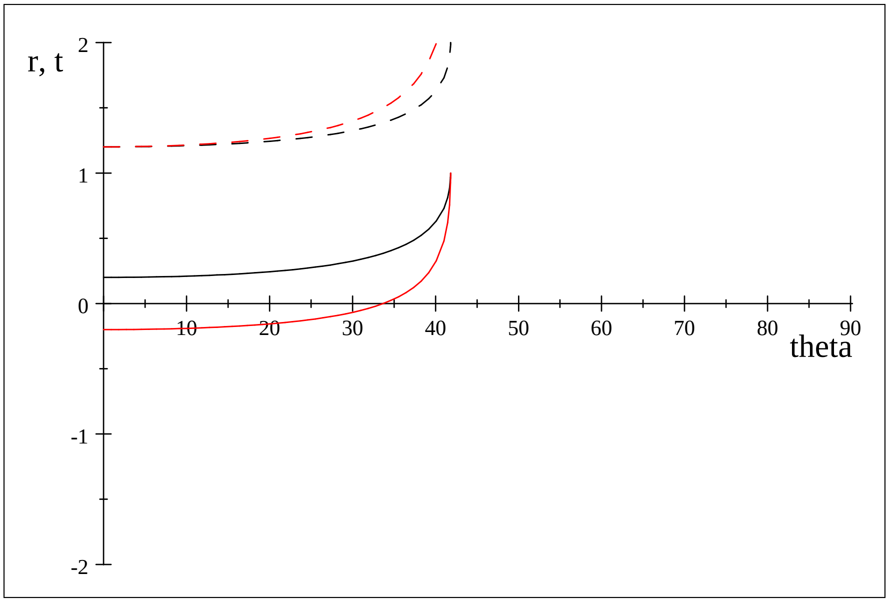
$$\theta_B = \tan^{-1}\left[\frac{1}{1.5}\right] \cong 0.588 \text{ radians} \cong 33.7^\circ$$

Critical Angle:

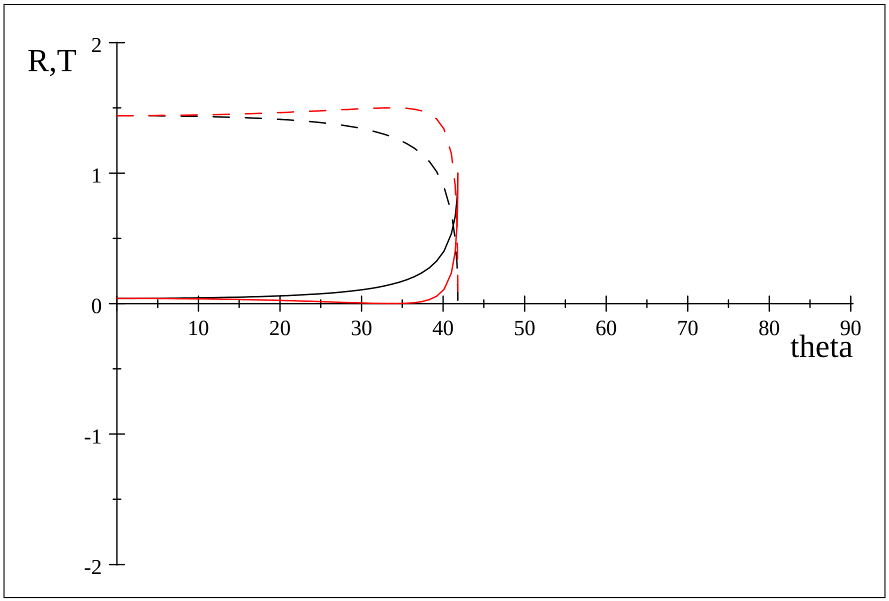
$$\theta_C = \sin^{-1}\left[\frac{1}{1.5}\right] \cong 0.730 \text{ radians} \cong 41.8^\circ$$

$$\begin{aligned} r_{TE} &= \frac{+1.5\cos[\theta_0] - 1\cdot\sqrt{1-\sin^2[\theta_0]}}{+1.5\cos[\theta_0] + 1\cdot\sqrt{1-\sin^2[\theta_0]}} \\ r_{TM} &= \frac{+1\cos[\theta_0] - 1.5\cdot\sqrt{1-\sin^2[\theta_0]}}{+1\cos[\theta_0] + 1.5\cdot\sqrt{1-\sin^2[\theta_0]}} \end{aligned}$$

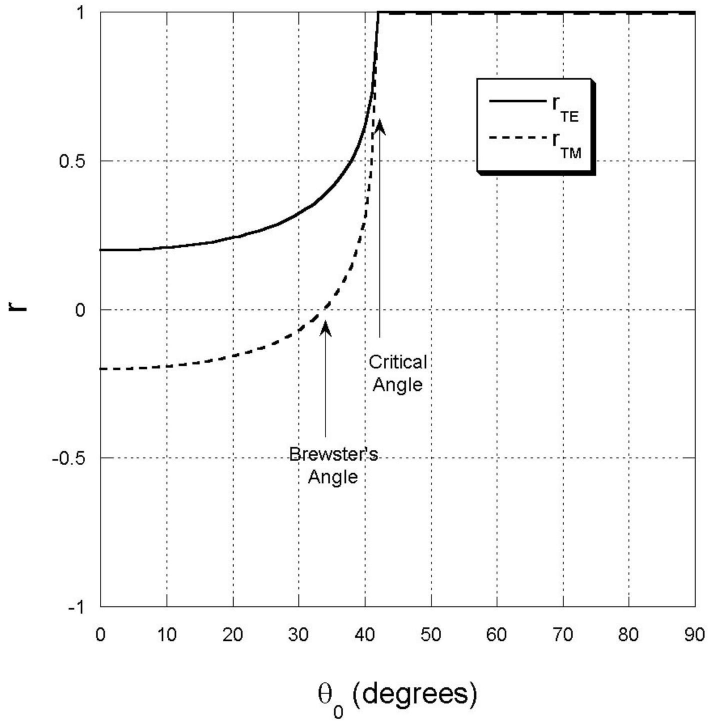
$$\begin{aligned} t_{TE} &= \frac{+2\cdot 1.5\cdot\cos[\theta_0]}{1.5\cdot\cos[\theta_0] + 1\cdot\cos[\theta_t]} \\ t_{TM} &= \frac{+2\cdot 1.5\cdot\cos[\theta_0]}{+1\cdot\cos[\theta_0] + 1.5\cdot\cos[\theta_t]} \end{aligned}$$



Amplitude coefficients r_{TE} (solid black), r_{TM} (dashed black), t_{TE} (solid red), and t_{TM} (dashed red); note that $r_{TM} = 0$ at Brewster's Angle and that $t_{TE} = t_{TM} = 0$ at $\theta_0 = \frac{\pi}{2}$ radians.



Reflectance and transmittance Rr_{TE} (solid black), R_{TM} (dashed black), T_{TE} (solid red), and T_{TM} (dashed red); note that $r_{TM} = 0$ at Brewster's Angle and that $t_{TE} = t_{TM} = 0$ at $\theta_0 = \frac{\pi}{2}$ radians.



Amplitude reflectance coefficients for TE and TM waves at a "dense-to-rare" interface with $n_1 = 1.5$ *(glass) and* $n_2 = 1.0$ *(air). Both polarizations rise to* $r = +1.0$ *at the "critical angle"* θ_c*, for which* $\theta_t = 90^\circ = \frac{\pi}{2}$*. Also noted is Brewster's angle, where* $r_{TM} = 0$*. The coefficients for* $\theta_0 > \theta_c$ *may be interpreted as being complex-valued.*

1.17 Practical Applications for Fresnel's Equations

- 4% reflectivity at normal incidence for one surface of glass
  - Windows resemble mirrors at night for persons in brightly lit rooms
- Windows at ends of gas tube in He:Ne laser oriented at Brewster's angle
  - Eliminate reflective losses
  - Emitted laser light is linearly polarized
- Basic principle of optical fiber transmission is total internal reflection to propagate beam