1. Consider monochromatic light of wavelength λ_0 incident on a single slit of width b along the x-axis and infinite length along y. The light is observed in the Fraunhofer diffraction region at a distance L. Derive an approximate expression for the *angular width at half-maximum irradiance* of the central peak of the diffraction spot (i.e., the FWHM).

   If observed in the Fraunhofer diffraction region, the observed irradiance is proportional to the squared magnitude of the appropriately scaled Fourier transform:

$$g[x,y;\lambda_0,z_1] \propto \frac{1}{z_1}\exp\left[+2\pi i\left(\frac{z_1}{\lambda_0}-\nu_0 t\right)\right]\cdot F\left[\xi=\frac{x}{\lambda_0 z_1},\eta=\frac{y}{\lambda_0 z_1}\right]$$

   The object function is:

$$f[x,y] = RECT\left[\frac{x}{b}\right]\cdot 1[y]$$

   Its Fourier transform is:

$$\begin{aligned} F[\xi,\eta] &= |b|\cdot SINC[b\xi]\cdot\delta[\eta] \\ &= |b|\cdot\frac{\sin[\pi b\xi]}{\pi b\xi}\cdot\delta[\eta] \end{aligned}$$

   where $\delta[\eta]$ is a 1-D Dirac delta function along the vertical direction, which has finite area and infinitesimal support along the vertical direction; we may think of $\delta[\eta]$ as concentrating all of the energy onto the x-axis. Substitute the variables into this expression:

$$\begin{aligned} F\left[\xi=\frac{x}{\lambda_0 z_1},\eta=\frac{y}{\lambda_0 z_1}\right] &= |b|\cdot\frac{\sin\left[\pi b\frac{x}{\lambda_0 z_1}\right]}{\pi b\frac{x}{\lambda_0 z_1}}\cdot\delta\left[\frac{y}{\lambda_0 z_1}\right] \\ &= \frac{\sin\left[\pi\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right]}{\left(\pi\frac{x}{\lambda_0 z_1}\right)}\cdot\delta\left[\frac{y}{\lambda_0 z_1}\right] \end{aligned}$$

   The irradiance is the square:

$$I[x,y]\propto|g[x,y]|^2\propto SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right]$$

   The full width at half maximum is the width of the $SINC^2$ function between the locations where its amplitude is $+\frac{1}{2}$:

$$\begin{aligned} SINC^2[u] = \frac{\sin^2[\pi u]}{(\pi u)^2} &= \frac{1}{2} \\ \Longrightarrow SINC[u] = \frac{\sin[\pi u]}{\pi u} &= \frac{1}{\sqrt{2}}\cong 0.7071 \end{aligned}$$

   The half angle $\theta_{\frac{1}{2}}$ is the value of u that satisfies this relation. We know that $SINC[0]=1$ and

$SINC[1]=0$. You can look up the appropriate value of u or solve for it by iteration:

$$\begin{aligned}
u &= 0.5 \Longrightarrow \frac{\sin\left[\frac{\pi}{2}\right]}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi} \cong 0.637 \\
u &= 0.4 \Longrightarrow \frac{\sin[0.4\pi]}{(0.4\pi)} \cong 0.757 \\
u &= 0.45 \Longrightarrow \frac{\sin[0.45\pi]}{(0.45\pi)} \cong 0.699 \\
u &= 0.44 \Longrightarrow \frac{\sin[0.44\pi]}{(0.44\pi)} \cong 0.711 \\
u &= 0.445 \Longrightarrow \frac{\sin[0.445\pi]}{(0.445\pi)} \cong 0.705 \\
u &= 0.4425 \Longrightarrow \frac{\sin[0.4425\pi]}{(0.4425\pi)} \cong 0.708
\end{aligned}$$

I'll call that close enough: $0.4425\pi \cong 1.390$ *radians*

$$\begin{aligned}
SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right] &= \frac{1}{2} \Longrightarrow \frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)} \cong 0.4425 \\
&\Longrightarrow x \cong 0.4425\frac{\lambda_0 z_1}{b} \\
&\Longrightarrow \theta_{\frac{1}{2}} \cong \frac{x}{z_1} \cong 0.4425\frac{\lambda_0}{b} \\
&\Longrightarrow \boxed{\theta = 2\cdot\theta_{\frac{1}{2}} \cong 0.885\frac{\lambda_0}{b}}
\end{aligned}$$

This often is rounded upward so that the full width at half maximum is approximately

$$\boxed{\theta \cong \frac{\lambda_0}{b}}$$

NOTE (again) that the width of the diffraction spot varies as the reciprocal of the slit width.

2. The Fraunhofer diffraction pattern of a pair of rectangular slits of width b separated by the center-to-center distance d is illuminated by monochromatic light with $\lambda_0 = 650\,\text{nm}$. The Fraunhofer diffraction pattern is viewed at the back focal plane of a lens with focal length $\mathbf{f} = 800\,\text{mm}$. The center-to=center separation between fringe maxima is observed to be $D = 1.04\,\text{mm}$. The fifth maximum of the interference pattern on each side is "missing," which means that it coincides with a zero in the diffraction "envelope" due to the width of the slit. Determine b and d.

The object function is:

$$f[x,y] = \left(RECT\left[\frac{x-\frac{d}{2}}{b}\right] + RECT\left[\frac{x+\frac{d}{2}}{b}\right]\right)\cdot 1[y]$$

The Fourier transform is:

$$\begin{aligned}
F[\xi,\eta] &= \left(\int_{\alpha=-\frac{d}{2}-\frac{b}{2}}^{\alpha=-\frac{d}{2}+\frac{b}{2}} \exp[-2\pi i\alpha\xi]\,d\alpha + \int_{\alpha=+\frac{d}{2}-\frac{b}{2}}^{\alpha=+\frac{d}{2}+\frac{b}{2}} \exp[-2\pi i\alpha\xi]\,d\alpha\right)\cdot\delta[\eta] \\
&= \frac{1}{-2\pi i\xi}\left(\exp\left[-2\pi i\left(-\frac{d}{2}+\frac{b}{2}\right)\xi\right] - \exp\left[-2\pi i\left(-\frac{d}{2}-\frac{b}{2}\right)\xi\right]\right)\cdot\delta[\eta] \\
&\quad + \frac{1}{-2\pi i\xi}\left(\exp\left[-2\pi i\left(+\frac{d}{2}+\frac{b}{2}\right)\xi\right] - \exp\left[-2\pi i\left(+\frac{d}{2}-\frac{b}{2}\right)\xi\right]\right)\cdot\delta[\eta] \\
&= \frac{1}{-2\pi i\xi}\exp\left[+2\pi i\left(\frac{d}{2}\right)\xi\right]\left(\exp\left[-2\pi i\left(\frac{b}{2}\right)\xi\right] - \exp\left[+2\pi i\left(\frac{b}{2}\right)\xi\right]\right)\cdot\delta[\eta] \\
&\quad + \frac{1}{-2\pi i\xi}\exp\left[-2\pi i\left(\frac{d}{2}\right)\xi\right]\left(\exp\left[-2\pi i\left(\frac{b}{2}\right)\xi\right] - \exp\left[+2\pi i\left(\frac{b}{2}\right)\xi\right]\right)\cdot\delta[\eta] \\
&= \frac{1}{\pi\xi}\exp[+i\pi d\xi]\sin[\pi b\xi]\cdot\delta[\eta] + \frac{1}{\pi\xi}\exp[-i\pi d\xi]\sin[\pi b\xi]\cdot\delta[\eta] \\
&= |b|\,SINC[b\xi]\cdot(\exp[+i\pi d\xi]+\exp[-i\pi d\xi])\cdot\delta[\eta] \\
&= 2|b|\,SINC[b\xi]\cdot\cos[\pi d\xi]\cdot\delta[\eta]
\end{aligned}$$

If observed at the Fraunhofer plane, the scaled Fourier transform is:

$$F\left[\xi = \frac{x}{\lambda_0 z_1}, \eta = \frac{y}{\lambda_0 z_1}\right] = 2|b|\,SINC\left[b\frac{x}{\lambda_0 z_1}\right]\cdot\cos\left[\pi d\frac{x}{\lambda_0 z_1}\right]\cdot\delta\left[\frac{y}{\lambda_0 z_1}\right]$$

Again, the Dirac delta function constrains the irradiance to the x-axis. The x-dependence of the irradiance is:

$$\begin{aligned}
I[x,y=0] &\propto 4|b|^2\,SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right]\cdot\cos^2\left[2\pi\frac{x}{\left(\frac{\lambda_0 z_1}{2d}\right)}\right] \\
&= 4|b|^2\,SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right]\cdot\frac{1}{2}\left(1+\cos\left[\pi\frac{x}{\left(\frac{\lambda_0 z_1}{d}\right)}\right]\right) \\
&= 2|b|^2\,SINC^2\left[\frac{x}{\left(\frac{\lambda_0 z_1}{b}\right)}\right]\cdot\left(1+\cos\left[\pi\frac{x}{\left(\frac{\lambda_0 z_1}{d}\right)}\right]\right)
\end{aligned}$$

In our situation, the light is observed at the back focal plane of a lens with focal length $\mathbf{f} = 800\,\text{mm}$, *so the The period of the observed cosine fringe is:*

$$D = \frac{\lambda_0\mathbf{f}}{d} = 1.04\,\text{mm} \Longrightarrow$$

$$\boxed{d = \frac{650\,\text{nm}\cdot 800\,\text{mm}}{1.04\,\text{mm}} = 0.5\,\text{mm}}$$

The fact that the fifth maximum is "missing" means that the $SINC^2$ *function decays to zero at that location:*

$$SINC^2\left[\frac{x}{\left(\frac{\lambda_0\mathbf{f}}{b}\right)}\right] = 0 \text{ if } b^{-1} = 5\cdot d^{-1} \Longrightarrow \boxed{b = \frac{d}{5} = 0.1\,\text{mm}}$$

3. Monochromatic light with wavelength λ_0 is incident on a circular aperture of diameter d. The circularly symmetric diffraction pattern observed at a distance L in the Fraunhofer diffraction region has radius r from the central maximum to the first zero:

$$r \cong 1.22\frac{L\lambda_0}{d}$$

(this distance is the separation required of the diffraction patterns from two point sources for them to be distinguished under the *Rayleigh criterion* for resolution)

(a) Compare this result to the linear distance from the central maximum to the first zero for a square aperture of width equal to the diameter d of the circular aperture.

The radius of the first zero of the diffraction pattern of the circular aperture is

$$r \cong 1.22\frac{L\lambda_0}{d}$$

The radius to the first zero of a square aperture is the value of x where:

$$SINC\left[\frac{x}{\left(\frac{L\lambda_0}{d}\right)}\right] = 0 \implies x = \boxed{\frac{L\lambda_0}{d} < 1.22\frac{L\lambda_0}{d}}$$

(b) If the monochromatic light at λ_0 illuminates a circular lens of diameter d and focal length $\mathbf{f}$, the Fraunhofer diffraction pattern is observed at the focal plane so that $L \cong \mathbf{f}$ and :

$$r \cong 1.22\frac{\mathbf{f}\lambda_0}{d}$$

If observed in blue visible light, find an approximate relation between the **diameter** of the diffraction spot and the focal length of the lens. This is a very convenient "rule of thumb" for imaging systems.

$$r \cong 1.22\frac{\mathbf{f}\lambda_0}{d} \implies 2r \cong 2.44\frac{\mathbf{f}\lambda_0}{d}$$

For blue light, $\lambda_0 \cong 400\,\text{nm} \implies 2.44\lambda_0$

$$\begin{aligned} &\implies && 2.44\lambda_0 \cong 1000\,\text{nm} \cong 1\,\mu\text{m} \\ 2r &\cong && 1\,\mu\text{m}\cdot\frac{\mathbf{f}}{d} \end{aligned}$$

We often define the focal ratio (f-number, f/#) of the lens as the ratio of the focal length to the diameter, so the diameter of the diffraction spot measured in micrometers is approximately equal to the f/# of the lens;

$$\boxed{2r \cong f/\#\,[\mu\text{m}]}$$

a smaller diameter lens leads to a larger diffraction spot.

4. Compare the diameters of the diffraction spots for telescopes with primary optics having diameters $d_1 = 200\,\text{in}$ (Hale Telescope on Palomar Mountain) and $d_2 = 90\,\text{mm}$ (Questar).

$$
\begin{aligned}
2r_{\text{Palomar}} &= 1\,\mu\text{m} \cdot \frac{\mathbf{f}}{200\,\text{in} \cdot \frac{1000\,\text{mm}}{39.37\,\text{in}}} \cong 1.97 \times 10^{-7} \cdot \mathbf{f} \\
2r_{\text{Questar}} &= 1\,\mu\text{m} \cdot \frac{\mathbf{f}}{90\,\text{mm}} \cong 1.11 \times 10^{-5} \cdot \mathbf{f}
\end{aligned}
$$

the two optics had the same focal length, the diffraction spot of the Palomar telescope would be smaller by about a factor of 56.

This is as far as you "had" to go, but you can go farther. The f/numbers of the two telescopes are very different; the Palomar primary mirror is f/3.3 $\Longrightarrow$ $\mathbf{f} = 3.3 \cdot d$, *while the focal length of the Questar is about* $1300\,\text{mm}$ $\Longrightarrow$ $f/\# = \frac{1300}{90} \cong 14$, *so the diffraction spots have radii approximately equal to* $3.3\,\mu\text{m}$ *for Palomar and* $14\,\mu\text{m}$ *for the Questar, which differ by a factor of about 4 instead of 56.*