

1. "White" light includes equal "amounts" of each wavelength in the interval  $400 \text{ nm} \leq \lambda \leq 700 \text{ nm}$ .

(a) Determine the frequency bandwidth for this wavelength range

$$\begin{aligned}\Delta\nu &\equiv |\nu_1 - \nu_2| = \left| \frac{c}{\lambda_1} - \frac{c}{\lambda_2} \right| \\ &= c \left| \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right| \\ &\cong 2.99792458 \times 10^8 \text{ m s}^{-1} \cdot \left| \frac{1}{400 \text{ nm}} - \frac{1}{700 \text{ nm}} \right| \\ &\quad \boxed{\Delta\nu \cong 3.212 \times 10^{14} \text{ Hz}}\end{aligned}$$

(b) Compute the associated *coherence time* and *coherence length* of white light.

$$\begin{aligned}\text{coherence time } \Delta t &\equiv \frac{1}{\Delta\nu} \cong \frac{1}{3.212 \times 10^{14} \text{ Hz}} \cong \boxed{3.113 \times 10^{-15} \text{ s}} \\ \text{coherence length } \ell &\equiv c \cdot \Delta t = \frac{c}{\Delta\nu} \cong 9.334 \times 10^{-7} \text{ m} \cong \boxed{93.34 \mu\text{m}}\end{aligned}$$

2. The range of angular temporal frequencies by a light source is  $\Delta\omega$ :

(a) Find an expression for  $\Delta\nu$

$$\omega = 2\pi\nu \implies \Delta\omega = 2\pi\Delta\nu \implies \boxed{\Delta\nu = \frac{\Delta\omega}{2\pi}}$$

(b) Derive an expression for the *linewidth*  $\Delta\lambda$

$$\begin{aligned}\lambda\nu &= c \implies \Delta c = 0 = \lambda \cdot \Delta\nu + \nu \cdot \Delta\lambda \\ \implies \frac{\Delta\nu}{\nu} &= -\frac{\Delta\lambda}{\lambda} \implies \Delta\lambda = -\frac{\lambda}{\nu} \Delta\nu = \boxed{-\frac{c}{\nu^2} \Delta\nu = \Delta\lambda}\end{aligned}$$

(c) Use the result of (a) to find an expression for the coherence length of the source.

$$\ell = \frac{c}{\Delta\nu} = \frac{2\pi c}{\Delta\omega}$$

(d) If the light source is a sodium arc that emits two narrow spectral lines with  $\omega_1 = 3.195 \times 10^{15} \frac{\text{radians}}{\text{sec}}$  and  $\omega_2 = 3.198 \times 10^{15} \frac{\text{radians}}{\text{sec}}$ , find the coherence length.

$$\begin{aligned}\ell &= \frac{2\pi c}{\Delta\omega} = 2\pi \frac{2.99792458 \times 10^8 \text{ m s}^{-1}}{|3.195 \times 10^{15} \frac{\text{radians}}{\text{sec}} - 3.198 \times 10^{15} \frac{\text{radians}}{\text{sec}}|} \\ &\quad \boxed{\ell \cong +62.8 \text{ mm}}\end{aligned}$$

(e) If the light source is a He:Ne "greenie" laser with  $\omega_1 = 3.171 \times 10^{15} \frac{\text{radians}}{\text{sec}}$  and  $\omega_2 = 3.469 \times 10^{15} \frac{\text{radians}}{\text{sec}}$ , find the coherence length.

$$\begin{aligned}\ell &= \frac{2\pi c}{\Delta\omega} = 2\pi \frac{2.99792458 \times 10^8 \text{ m s}^{-1}}{|3.171 \times 10^{15} \text{ s}^{-1} - 3.469 \times 10^{15} \text{ s}^{-1}|} \\ &\quad \boxed{\ell \cong +6.32 \mu\text{m}} \text{ much shorter because } \Delta\lambda \text{ is much larger}\end{aligned}$$

3. Determine the linewidth in nanometers and in Hertz for laser light whose coherence length is 10 km if the mean wavelength is 632.8 nm (He:Ne)

$$\begin{aligned} \ell &= \frac{c}{\Delta\nu} \implies \Delta\nu = \frac{c}{\ell} \\ \ell &= 10 \text{ km} \implies \Delta\nu \cong \frac{2.99792458 \times 10^8 \text{ m s}^{-1}}{10 \text{ km}} \cong \boxed{29.979 \text{ kHz} \cong 3 \times 10^4 \text{ Hz} \cong \Delta\nu} \\ \Delta\nu &= \\ \Delta\lambda &= \left| -\frac{c}{\nu_0^2} \Delta\nu \right| = \frac{c}{\left(\frac{c}{\lambda_0}\right)^2} \cdot \Delta\nu = \frac{\lambda_0^2}{c} \cdot \Delta\nu \\ &\cong \frac{(632.8 \text{ nm})^2}{2.99792458 \times 10^8 \text{ m s}^{-1}} \cdot 29.979 \text{ kHz} \\ &\quad \boxed{\Delta\lambda \cong 4.00 \times 10^{-17} \text{ m} = 4.00 \times 10^{-8} \text{ nm}} \end{aligned}$$

4. Michelson found that the cadmium red line ( $\lambda_0 = 643.8 \text{ nm}$ ) was the best available light source for his interference experiment. With it, he could see fringes for optical path differences up to 300 mm. Estimate the linewidth  $\Delta\lambda$  and coherence time  $\Delta t$  of this light source.

$$\begin{aligned} OPD &= 300 \text{ mm} = \ell = c \cdot \Delta t = \frac{c}{\Delta\nu} \\ \implies \Delta\nu &= \frac{c}{\ell} \cong \frac{2.99792458 \times 10^8 \text{ m s}^{-1}}{300 \text{ mm}} \cong 10^9 \text{ Hz} = 1000 \text{ MHz} \\ \Delta t &= \frac{1}{\Delta\nu} = 10^{-9} \text{ s} = \boxed{1 \text{ ns} = \Delta t} \\ \Delta\lambda &= \frac{\lambda_0^2}{c} \cdot \Delta\nu = \frac{(643.8 \text{ nm})^2}{2.99792458 \times 10^8 \text{ m s}^{-1}} \cdot 10^9 \text{ Hz} \cong 1.382 \times 10^{-12} \text{ m} \\ &\quad \boxed{\Delta\lambda = 1.382 \times 10^{-3} \text{ nm}} \end{aligned}$$

5. A light source emits two wavelengths  $\lambda_1$  and  $\lambda_2$ . The light is incident upon a binary object (composed of regions that are perfectly transparent or perfectly opaque)  $f[x, y]$ . The light then propagates to an observation screen located at a very large distance  $L$  from the object. Describe and give reasons for the *qualitative* appearance of the observed patterns for the following objects; you may also describe the patterns quantitatively for extra credit.

- (a)  $f_a[x, y]$  is a single very small transparent aperture (“hole”)

*One clue is the phrase “very large distance  $L$ ” which implies Fraunhofer diffraction region, so the observed amplitude is proportional to sums of patterns at the two wavelengths. If the aperture is very small, then its Fourier transform is very large – the Fourier transform of an aperture with infinitesimal support (Dirac delta function) has infinite support. If we use that as the model, the diffraction pattern will include equal weights of both colors.*

- (b)  $f_b[x, y]$  consists of two apertures that are very narrow along the  $x$ -axis and infinitely long along the  $y$ -axis and that are separated by  $d$  units.

*This is the two-aperture problem. For each color, we know that the diffracted amplitude is a cosine fringe:*

$$g[x, y; \lambda_1] = \frac{1}{z_1} \exp \left[ 2\pi i \left( \frac{z_1}{\lambda_1} - \nu_1 t \right) \right] \cdot \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_1 z_1}{2d} \right)} \right]$$

$$g[x, y; \lambda_2] = \frac{1}{z_1} \exp \left[ 2\pi i \left( \frac{z_1}{\lambda_2} - \nu_2 t \right) \right] \cdot \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_2 z_1}{2d} \right)} \right]$$

*The mix is the sum of the the two patterns, which will be*

$$g[x, y; \lambda_1] + g[x, y; \lambda_2] = \frac{1}{z_1} \left( \exp \left[ 2\pi i \left( \frac{z_1}{\lambda_1} - \nu_1 t \right) \right] \cdot \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_1 z_1}{2d} \right)} \right] + \exp \left[ 2\pi i \left( \frac{z_1}{\lambda_2} - \nu_2 t \right) \right] \cdot \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_2 z_1}{2d} \right)} \right] \right)$$

*The irradiance is approximately equal to the sum of two cosine<sup>2</sup>-functions:*

$$|g[x, y; \lambda_1] + g[x, y; \lambda_2]|^2 \cong \frac{1}{z_1^2} \left( \cos^2 \left[ 2\pi \frac{x}{\left( \frac{\lambda_1 z_1}{2d} \right)} \right] + \cos^2 \left[ 2\pi \frac{x}{\left( \frac{\lambda_2 z_1}{2d} \right)} \right] \right)$$

$$= \frac{1}{z_1^2} \left( 1 + \frac{1}{2} \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_1 z_1}{d} \right)} \right] + \frac{1}{2} \cos \left[ 2\pi \frac{x}{\left( \frac{\lambda_2 z_1}{d} \right)} \right] \right)$$

*The two cosine functions have different periods:*

$$D_1 = \frac{\lambda_1 z_1}{d}$$

$$D_2 = \frac{\lambda_2 z_1}{d}$$

*The maxima coincide at the center ( $x = 0$ ), but the first-order maximum at the longer wavelength is at a larger value of  $x$ .*

- (c)  $f_c[x, y]$  consists of an infinite number of apertures from part *b* that are uniformly spaced at increments of  $d$  units.

*As more apertures are added, the “widths” of the cosine fringes get narrower but the periods remain the same. With many apertures, the patterns become “sharp” with the same periods  $D_1$  and  $D_2$  so you can see the light from the two wavelengths separated in locations; this is a diffraction grating.*

6. The light diffracted by an object of the form  $f[x, y]$  and observed at a distance  $z_1$  in the Fraunhofer diffraction region has the “shape” of the squared magnitude of the Fourier transform of the object after appropriate rescaling of the coordinates back to the space domain

$$g[x, y] \propto |F[\xi, \eta]|^2 \Big|_{\xi \rightarrow \frac{x}{\lambda_0 z_1}, \eta \rightarrow \frac{y}{\lambda_0 z_1}}$$

where the 2-D Fourier transform is defined:

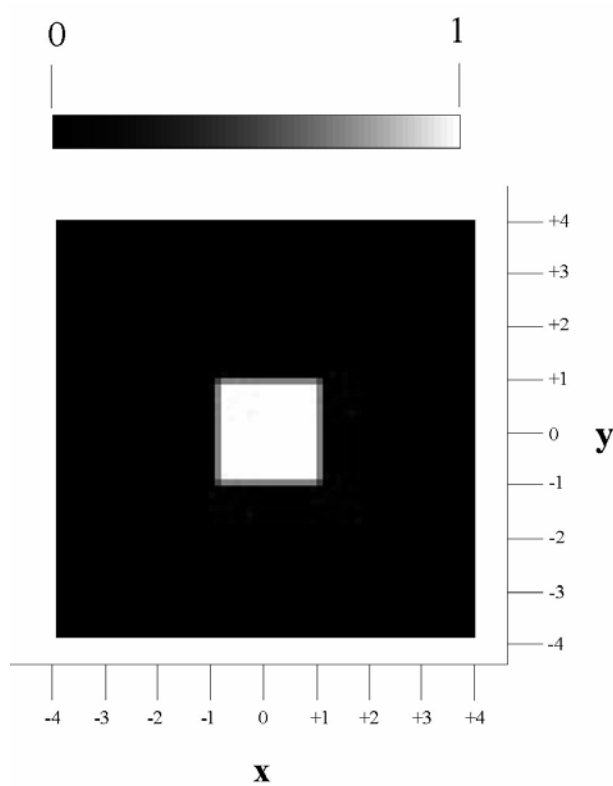
$$F[\xi, \eta] \equiv \mathcal{F}_2 \{f[x, y]\} \equiv \iint_{-\infty}^{+\infty} f[x, y] \exp[-2\pi i(\xi x + \eta y)] dx dy$$

The object  $f[x, y]$  satisfies the following conditions:

$$\begin{aligned} f[x, y] &= 1 && \text{if } |x| \leq 1 \text{ AND } |y| \leq 1 \\ f[x, y] &= 0 && \text{if otherwise} \end{aligned}$$

- (a) Sketch  $f[x, y]$ ;

*This is a rectangle function of width 2 units in each direction*



- (b) Calculate the diffraction pattern in the Fraunhofer diffraction region if  $f[x, y]$  is illuminated by light with wavelength  $\lambda_0$ ;

*The observation plane is in the Fraunhofer diffraction region, so the output is approximately:*

$$\begin{aligned}
g[x, y] &\cong \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \iint_{-\infty}^{+\infty} \text{RECT} \left[ \frac{\alpha}{2}, \frac{\beta}{2} \right] \exp \left[ -2\pi i \left( \alpha \cdot \frac{x}{\lambda_0 z_1} + \beta \cdot \frac{y}{\lambda_0 z_1} \right) \right] d\alpha d\beta \\
&= \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \int_{-\infty}^{+\infty} \text{RECT} \left[ \frac{\alpha}{2} \right] \exp \left[ -2\pi i \alpha \cdot \frac{x}{\lambda_0 z_1} \right] d\alpha \right) \\
&\quad \cdot \left( \int_{-\infty}^{+\infty} \text{RECT} \left[ \frac{\beta}{2} \right] \exp \left[ -2\pi i \beta \cdot \frac{y}{\lambda_0 z_1} \right] d\beta \right) \\
&= \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \int_{-1}^{+1} 1 \cdot \exp \left[ -2\pi i \alpha \cdot \frac{x}{\lambda_0 z_1} \right] d\alpha \right) \\
&\quad \cdot \left( \int_{-1}^{+1} 1 \cdot \exp \left[ -2\pi i \beta \cdot \frac{y}{\lambda_0 z_1} \right] d\beta \right) \\
&= \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \frac{1}{-2\pi i \frac{x}{\lambda_0 z_1}} \exp \left[ -2\pi i \alpha \cdot \frac{x}{\lambda_0 z_1} \right] \Big|_{\alpha=-1}^{\alpha=+1} \right) \\
&\quad \cdot \left( \frac{1}{-2\pi i \frac{y}{\lambda_0 z_1}} \exp \left[ -2\pi i \beta \cdot \frac{y}{\lambda_0 z_1} \right] \Big|_{\beta=-1}^{\beta=+1} \right) \\
&= \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \frac{1}{-2\pi i \frac{x}{\lambda_0 z_1}} \left( \exp \left[ -2\pi i \cdot \frac{x}{\lambda_0 z_1} \right] - \exp \left[ +2\pi i \cdot \frac{x}{\lambda_0 z_1} \right] \right) \right) \\
&\quad \cdot \left( \frac{1}{-2\pi i \frac{y}{\lambda_0 z_1}} \left( \exp \left[ -2\pi i \cdot \frac{y}{\lambda_0 z_1} \right] - \exp \left[ +2\pi i \cdot \frac{y}{\lambda_0 z_1} \right] \right) \right) \\
&= \frac{1}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \frac{1}{+\frac{\pi x}{\lambda_0 z_1}} \left( \sin \left[ +2\pi \frac{x}{\lambda_0 z_1} \right] \right) \right) \left( \frac{1}{+\frac{\pi y}{\lambda_0 z_1}} \left( \sin \left[ +2\pi \frac{y}{\lambda_0 z_1} \right] \right) \right) \\
&= \frac{4}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \frac{1}{+\frac{2\pi x}{\lambda_0 z_1}} \left( \sin \left[ +2\pi \frac{x}{\lambda_0 z_1} \right] \right) \right) \left( \frac{1}{+\frac{2\pi y}{\lambda_0 z_1}} \left( \sin \left[ +2\pi \frac{y}{\lambda_0 z_1} \right] \right) \right) \\
&= \frac{4}{z_1} \exp \left[ +2\pi i \left( \frac{z_1}{\lambda_0} - \nu_0 t \right) \right] \left( \text{SINC} \left[ +\frac{x}{\left( \frac{\lambda_0 z_1}{2} \right)} \right] \right) \left( \text{SINC} \left[ +\frac{y}{\left( \frac{\lambda_0 z_1}{2} \right)} \right] \right)
\end{aligned}$$

*The observed irradiance is:*

$$|g[x, y]|^2 \cong \frac{16}{|z_1|^2} \text{SINC}^2 \left[ \frac{x}{\left( \frac{\lambda_0 z_1}{2} \right)}, \frac{y}{\left( \frac{\lambda_0 z_1}{2} \right)} \right]$$

(c) Sketch the x-axis profile of the diffraction pattern including labels of the values on both axes.

