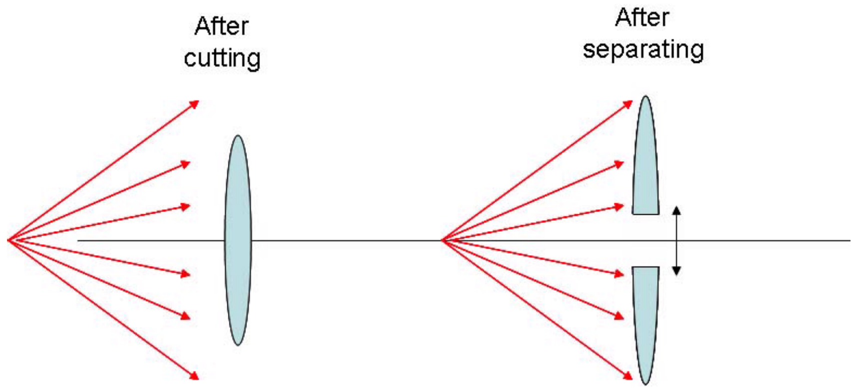
SIMG-455 Solution Set #5

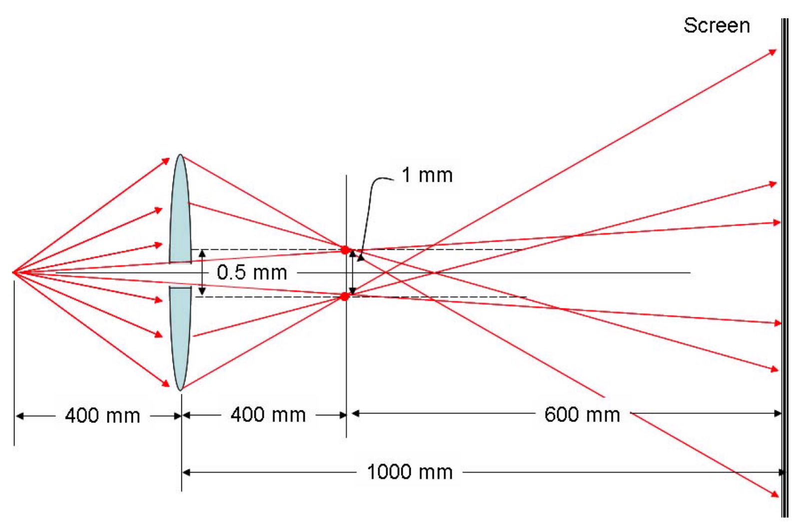
1. A lens with $f = +200\,\text{mm}$ is sawn in two pieces through a plane cutting through the optical axis (i.e., the cut is along a diameter). A point source S of monochromatic light with $\lambda = 500\,\text{nm}$ is placed on the optical axis at a distance $z_1 = 400\,\text{mm}$ from the lens. The half lenses are gradually moved apart; each creates an image of the point source that are mutually coherent. The light is observed on a screen placed at a distance $z_2 = 1000\,\text{mm}$ from the lens. Determine the width of the interference fringes observed on the screen if the lenses are moved apart by a distance of $0.5\,\text{mm}$. ((Hint, problem is easy if you look at it correctly)



The lens creates two images of the point source. We know from the imaging equation that:

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{\mathbf{f}} \Longrightarrow z_2 = \frac{z_1 \mathbf{f}}{z_1 - \mathbf{f}} = \frac{400\,\text{mm} \cdot 200\,\text{mm}}{400\,\text{mm} - 200\,\text{mm}} = 400\,\text{mm}$$

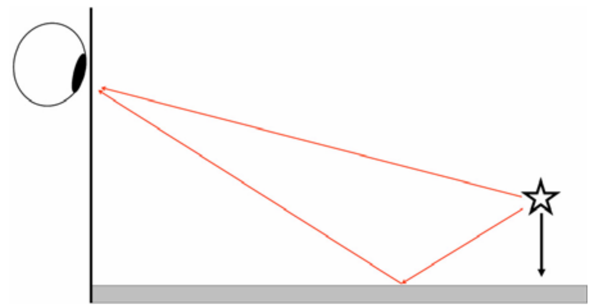
which you probably knew already



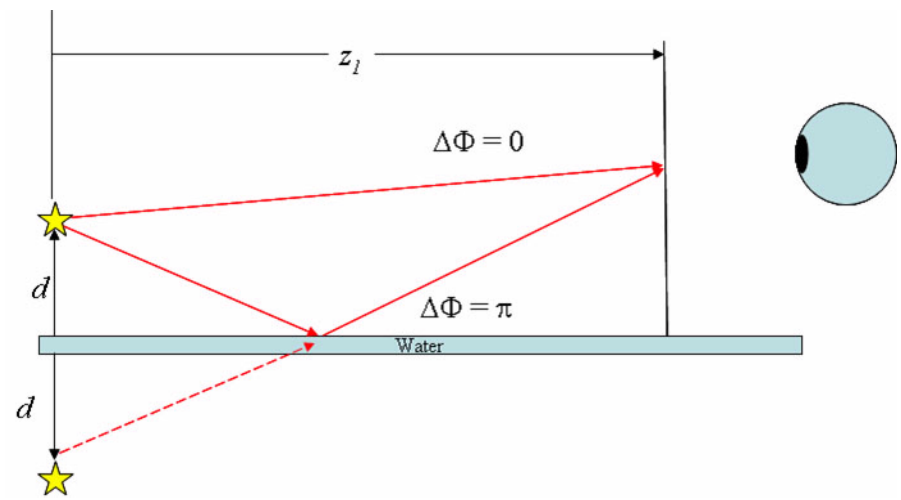
From sketch, we see that $d = 1\,\text{mm}$ *and* $L = 600\,\text{mm} \Longrightarrow$

$$D = \frac{L\lambda}{d} = \frac{600 \times 10^{-3}\,\text{m} \cdot 500 \times 10^{-9}\,\text{m}}{1 \cdot 10^{-3}\,\text{m}} = \boxed{0.3\,\text{mm} = D}$$

2. A monochromatic star with $\lambda_0 = 460\,\text{nm}$ is setting over a smooth ocean surface. Assume that we are located at the equator and the star is on the celestial equator, so that the star path is perpendicular to the ocean surface. Some of the starlight travels directly to the observer and some is reflected from the surface. Describe qualitatively and quantitatively what is observed at the observation plane as a function of both space and time; include a sketch of the pattern observed at one time and indicate what happens as the star approaches the horizon. Possibly useful information: the earth rotates 360° in 24 hours, so the star appears to move 15° in one hour.



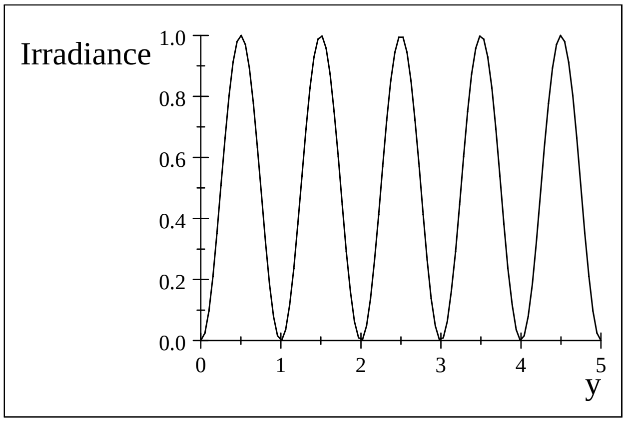
This is a disguised two-aperture interference experiment, as may perhaps be more easily seen if I redraw the system:



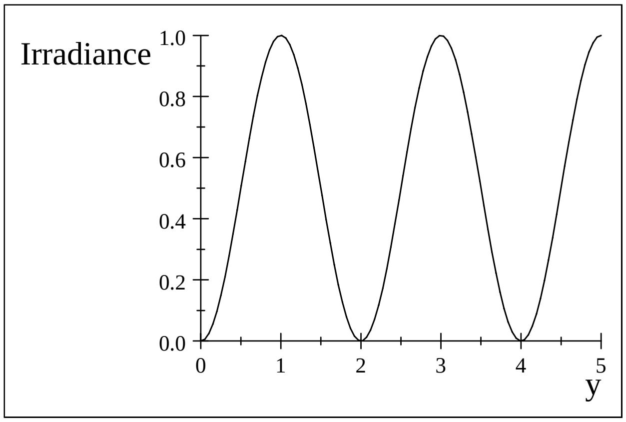
There is a phase change of π radians due to the reflection, which means that there is destructive interference at the locations where the two paths are equal modulo λ_0, as at the intersection between the observation screen and the water surface. The fringe pattern has exactly the same period as in the two-aperture experiment:

$$D_0 = \frac{z_1 \cdot \lambda_0}{d}$$

Since the earth is rotating, the star moves closer to the horizon as time increases as the star is about to set; since we're at the equator, the star motion is perpendicular to the horizon. The earth moves 15° in one hour, or 1° in four minutes. The period of the observed fringe pattern would increase as the star approaches the horizon, but the interference minimum would always be on the horizon.



Normalized irradiance pattern as function of "height" at one time t_0

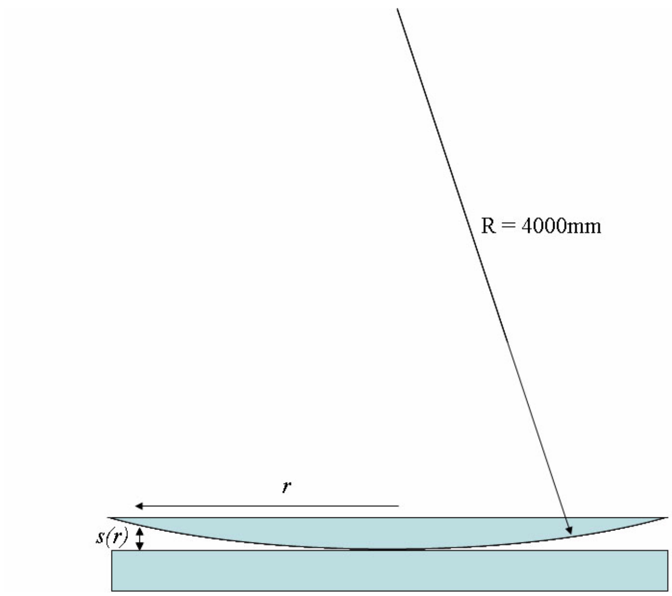


Normalized irradiance pattern as function of "height" at later time $t_1 > t_0$ *as the star approaches the horizon.*

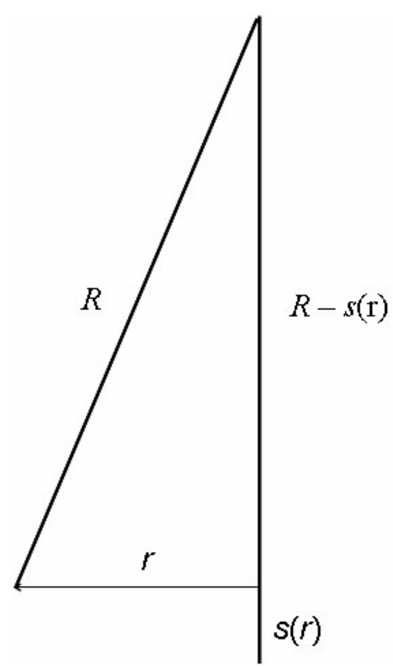
3. *Newton's rings* is the name given to the interference pattern observed when a plano-convex lens is placed (convex side down) on an optical flat and illuminated by mono-chromatic (or nearly monochromatic) light. In one such example the diameter of the first bright is 2 mm.

   (a) If the radius of curvature of the convex surface is 4 m, then determine the wave-length λ_0 of the illumination.

   The light that reflects from the "inside" of the upper surface interferes with the reflection from the "outside" of the lower surface. The primary task here is to find the distance between the two surfaces. For the spherical and planar surfaces, the thickness of the "air gap" between the two is called the "sagitta" or "sag":



   The sag formula may be found from the sketch:



$$\begin{aligned}(R-s[r])^2+r^2 &= R^2\\ &\Longrightarrow R^2+s^2[r]-2R\cdot s[r]+r^2=R^2\\ &\Longrightarrow s^2[r]-2R\cdot s[r]+r^2=0\end{aligned}$$

If $s[r] << R \Longrightarrow s^2[r] \cong 0:$

$$\begin{aligned}-2R\cdot s[r]+r^2 &\cong 0\\ &\Longrightarrow s[r]\cong\frac{r^2}{2R}\end{aligned}$$

The optical path difference between the light reflected from the top surface and that from the bottom surface is twice the thickness of the gap:

$$OPD \cong 2\cdot\frac{r^2}{2R}=\frac{r^2}{R}$$

The reflection at the bottom surface has a phase change of π *radians (rare-to-dense), so the optical phase difference is:*

$$\begin{aligned}O\Phi D &\cong \frac{2\pi}{\lambda_0}\cdot\frac{r^2}{R}+\pi\\ &= 2\pi\left(\frac{r^2}{\lambda_0 R}+\frac{1}{2}\right)\end{aligned}$$

There will be an interference maximum if the phase difference is an even integer multiple of π *and a minimum if the phase difference is an odd multiple of* π*. If* $r=0$ *(center of the lens), the phase difference is* π *in reflection* $\Longrightarrow$ *interference minimum. The first maximum occurs where:*

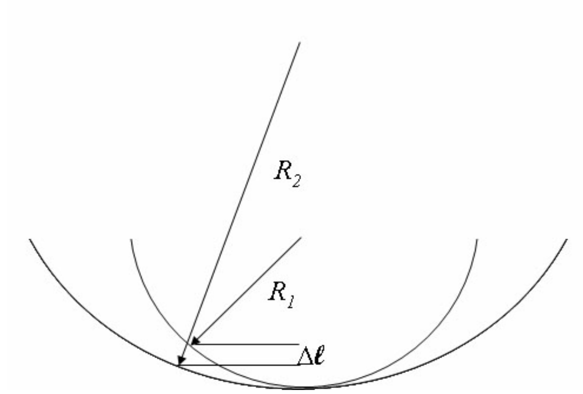
$$\begin{aligned}2\pi\left(\frac{r^2}{\lambda_0 R}+\frac{1}{2}\right) &= 2\pi \Longrightarrow \frac{r^2}{\lambda_0 R}+\frac{1}{2}=1\\ &\Longrightarrow \frac{r^2}{\lambda_0 R}=\frac{1}{2}\\ &\Longrightarrow \lambda_0=\frac{2r^2}{R}=\frac{2\cdot(1\,\text{mm})^2}{4000\,\text{mm}}=\boxed{500\,\text{nm}=\lambda_0}\end{aligned}$$

(b) If the space between the glass surfaces is filled with water $\left(n=1.33\cong\frac{4}{3}\right)$, determine the diameter of the first bright ring.

The wavelength of light in water is scaled by the factor of $\frac{1}{n}$ *and so is approximately* $\frac{3}{4}$ *times as long, so the gap between the surfaces includes more wavelengths in water, and thus the optical phase changes "faster"*

$$
\begin{aligned}
O\Phi D &\cong \frac{2\pi}{\left(\frac{\lambda_0}{n}\right)} \cdot \frac{r^2}{R} + \pi \\
&= 2\pi \left(\frac{nr^2}{\lambda_0 R} + \frac{1}{2} \right) \\
2\pi \left(\frac{nr^2}{\lambda_0 R} + \frac{1}{2} \right) &= 2\pi \implies \frac{nr^2}{\lambda_0 R} + \frac{1}{2} = 1 \\
\frac{1.33 \cdot r^2}{500\,\mathrm{nm} \cdot 4000\,\mathrm{mm}} &= \frac{1}{2} \implies r = \sqrt{\frac{1}{2} \cdot \frac{500\,\mathrm{nm} \cdot 4000\,\mathrm{mm}}{1.33}} \cong 0.867\,\mathrm{mm} \\
& \boxed{\textit{diameter } d = 2r \cong 1.734\,\mathrm{mm}} \\
& \textit{which is smaller than the diameter if the lenses are in air}
\end{aligned}
$$

4. (OPS 9-33) The radius of curvature of the convex surface of a plano-convex lens is $r_1 = 2\,\mathrm{m}$. The lens is placed convex side down on the *concave* surface of a plano-concave lens with $r_2 = 4\,\mathrm{m}$. The lenses are illuminated from above with light with $\lambda_0 = 625\,\mathrm{nm}$.



(a) Describe the pattern observed; a sketch would be helpful;

The optical path difference is the difference of the sags:

$$s_1(r) \cong \frac{r^2}{2R_1}$$
$$s_2(r) \cong \frac{r^2}{2R_2}$$

$$s_1(r) - s_2(r) = \frac{r^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = OPD$$
$$O\Phi D = \frac{2\pi}{\lambda_0}\left(s_1(r) - s_2(r)\right) + \pi$$
$$= \pi\left[\frac{r^2}{\lambda_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + 1\right]$$

(b) Find the diameter of the third bright ring in the interference pattern of the reflected light.

Third bright ring means that the optical phase difference from the center is $O\Phi D = 3 \cdot 2\pi + \pi = 7\pi$:

$$\pi\frac{r^2}{\lambda_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + 1 = 7\pi$$
$$\frac{r^2}{625\,\mathrm{nm}}\left(\frac{1}{2\,\mathrm{m}} - \frac{1}{4\,\mathrm{m}}\right) = \frac{r^2}{625\,\mathrm{nm}} \cdot \frac{1}{4\,\mathrm{m}} = 6$$
$$r^2 = 6 \cdot 625\,\mathrm{nm} \cdot 4\,\mathrm{m} = 1.5 \times 10^{-5}\,\mathrm{m}^2$$
$$\boxed{r = \sqrt{1.5 \times 10^{-5}\,\mathrm{m}^2} \cong 3.87\,\mathrm{mm}}$$

5. A two-aperture apparatus may be used to measure the index of refraction of gases. The apertures are separated by d and illuminated from the left by monochromatic plane waves with wavelength λ_0. The resulting pattern is observed on a screen located a distance L from the aperture plane. Two identical glass containers are placed just after the apertures; the lengths of both chambers along the axis of symmetry are identically ℓ.

   (a) With both chambers evacuated (i.e., both are "filled with vacuum"), describe what is observed on the screen.

   You get the standard two-aperture pattern; sinusoidal fringes of the form:

$$I[x] \propto 1 + \cos\left[2\pi \frac{x}{D}\right]; \quad D \cong \frac{L\lambda_0}{d}$$

   (b) If a gas is admitted into one of the two chambers, describe what happens to the pattern; be specific about any directions involved.

   If gas is admitted into one chamber, the optical path length through that chamber must increase due to the index of refraction increasing. The "central" fringe of the pattern where the optical paths match must move to maintain equal optical path, therefore the path through the other aperture must increase to match; the fringes move in the direction of the chamber with the gas.

   (c) Use the results of (a) and (b) to find an expression for the refractive index n of the gas.

   If the path through the chamber is ℓ units long and if the pattern moves by p units towards the direction of the chamber with the gas, then we know that the optical path length through the "open" aperture has increased

$$\sqrt{\left(\frac{d}{2}\right)^2 + L^2} \to \sqrt{\left(\frac{d}{2} + p\right)^2 + L^2}$$

   so the optical path through the aperture with the gas must match:

$$\begin{aligned} L &\to n \cdot \ell + 1 \cdot (L - \ell) \\ \sqrt{\left(\frac{d}{2}\right)^2 + L^2} &\to \sqrt{\left(\frac{d}{2}\right)^2 + (n \cdot \ell + 1 \cdot (L - \ell))^2} \\ &= \sqrt{\left(\frac{d}{2}\right)^2 + ((n-1) \cdot \ell + L)^2} \end{aligned}$$

$$
\begin{aligned}
\sqrt{\left(\frac{d}{2}\right)^2 + ((n-1)\cdot\ell + L)^2} &= \sqrt{\left(\frac{d}{2}+p\right)^2 + L^2} \\
\left(\frac{d}{2}\right)^2 + (n-1)^2\cdot\ell^2 + L^2 + 2\cdot(n-1)\,\ell\cdot L &= \left(\frac{d}{2}+p\right)^2 + L^2 \\
(n-1)^2\cdot\ell^2 + 2\cdot(n-1)\,\ell\cdot L &= p^2 + dp \cong p^2 \\
2\cdot(n-1)\,\ell\cdot L &\cong p^2 + dp \\
& \boxed{n \cong 1 + \frac{p\,(p+d)}{2L\ell}} \\
p &= 0 \implies n = 1 \textit{ (as it should)}
\end{aligned}
$$