1. The variation of refractive index with wavelength for a transparent substance (such as glass) may be approximately represented by the empirical equation due to Cauchy:

$$n[\lambda] \cong A_0 + \frac{B_0}{\lambda_0^2}$$

where A_0 and B_0 are empirically determined constants and λ_0 is the wavelength of light in a vacuum. If $A_0 = 1.40$, $B_0 = 2.5 \cdot 10^4\,(\mathrm{nm})^2$, determine the phase and group velocities at $\lambda_0 = 500\,\mathrm{nm}$.

$$n[\lambda = 500\,\mathrm{nm}] = 1.40 + \frac{2.5 \cdot 10^4\,(\mathrm{nm})^2}{(500\,\mathrm{nm})^2} = 1.5$$

$$\mathrm{v}_\phi[\lambda = 500\,\mathrm{nm}] = \frac{\lambda_0 \nu_0}{n} = \frac{\omega_0}{|k_0|} = \frac{c}{n} \cong \frac{2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}}{1.5} \cong \boxed{\mathrm{v}_\phi \cong 2.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}}$$

The "group velocity" may also be called the "modulation velocity" $\boldsymbol{v}_{\mathrm{mod}}$, *which is the speed of the low-frequency modulation. We need to find an expression for the modulation velocity in terms of the numbers we know:*

$$\mathrm{v}_{\mathrm{mod}} = \left.\frac{d\omega}{dk}\right|_{\lambda = 500\,\mathrm{nm}}$$

$$\mathrm{v}_\phi = \frac{\omega}{k} \Longrightarrow \omega = k \cdot \mathrm{v}_\phi = k \cdot \frac{c}{n}$$

$$\Longrightarrow \frac{d\omega}{dk} = \frac{d}{dk}\left(k \cdot \frac{c}{n}\right) = \frac{c}{n} \cdot \frac{dk}{dk} + \frac{k}{n} \cdot \frac{dc}{dk} + kc \cdot \frac{d}{dk}\left(\frac{1}{n}\right)$$

$$\mathrm{v}_{\mathrm{mod}} = \frac{d\omega}{dk} = \frac{c}{n} \cdot 1 + \frac{k}{n} \cdot 0 + kc \cdot \left(-n^{-2}\right)\frac{dn}{dk} = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{dk} = \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

$$\frac{dn}{dk} = \frac{dn}{d\lambda} \cdot \frac{d\lambda}{dk} = \frac{dn}{d\lambda} \cdot \left(\frac{dk}{d\lambda}\right)^{-1} = \frac{dn}{d\lambda} \cdot \left(-\frac{2\pi}{\lambda^2}\right)^{-1} = \frac{dn}{d\lambda} \cdot \left(-\frac{\lambda^2}{2\pi}\right)$$

$$\mathrm{v}_{\mathrm{mod}} = \mathrm{v}_\phi\left(1 - \frac{2\pi}{n\lambda} \cdot \frac{dn}{d\lambda} \cdot \left(-\frac{\lambda^2}{2\pi}\right)\right) = \mathrm{v}_\phi\left(1 + \frac{\lambda}{n} \cdot \frac{dn}{d\lambda}\right) = \frac{c}{n}\left(1 + \frac{\lambda}{n} \cdot \frac{dn}{d\lambda}\right)$$

$$n[\lambda] = A_0 + \frac{B_0}{\lambda^2} \Longrightarrow \frac{dn}{d\lambda} = -2\frac{B_0}{\lambda^3}$$

$$\mathrm{v}_{\mathrm{mod}} = \frac{c}{n}\left(1 + \frac{\lambda}{n} \cdot \left(-2\frac{B_0}{\lambda^3}\right)\right) = \frac{c}{n}\left(1 - 2 \cdot \frac{B_0}{n\lambda^2}\right)$$

$$= \frac{c}{1.5}\left(1 - 2 \cdot \frac{2.5 \cdot 10^4\,(\mathrm{nm})^2}{1.5 \cdot (500\,\mathrm{nm})^2}\right) = \frac{c}{1.7308} = \frac{\mathrm{v}_\phi}{1.154}$$

2. For the crown and flint glasses given in the notes with the following indices measured at two vacuum wavelengths:

Line	λ_0 [nm]	n for Crown	n for Flint
C	656.28	1.51418	1.69427
F	486.13	1.52225	1.71748

Approximate the empirical constants A_0 and B_0 in Cauchy's equation (given in #1) and use them to evaluate the refractive index at $\lambda_0 = 589.59\,\text{nm}$ (Fraunhofer's "D" line); compare the results to the actual values:

Line	λ_0 [nm]	n for Crown	n for Flint
D	589.59	1.51666	1.70100

Two equations in two unknowns:

$$n_C = A_0 + \frac{B_0}{\lambda_C^2}$$

$$n_F = A_0 + \frac{B_0}{\lambda_F^2}$$

Many ways to solve; I'll use matrix inversion.

For the crown glass:

$$\begin{bmatrix} 1 & \lambda_C^{-2} \\ 1 & \lambda_F^{-2} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} n_C \\ n_F \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & \lambda_C^{-2} \\ 1 & \lambda_F^{-2} \end{bmatrix}^{-1} \begin{bmatrix} n_C \\ n_F \end{bmatrix} = \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \lambda_C^{-2} \\ 1 & \lambda_F^{-2} \end{bmatrix} = \begin{bmatrix} 1 & (656.28\,\text{nm})^{-2} \\ 1 & (486.13\,\text{nm})^{-2} \end{bmatrix} = \begin{bmatrix} 1 & 2.321\,8 \times 10^{-6}\,\text{nm}^{-2} \\ 1 & 4.231\,5 \times 10^{-6}\,\text{nm}^{-2} \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 2.321\,8 \times 10^{-6}\,\text{nm}^{-2} \\ 1 & 4.231\,5 \times 10^{-6}\,\text{nm}^{-2} \end{bmatrix} = 1.9097 \times 10^{-6}\,\text{nm}^{-2}$$

$$\begin{bmatrix} 1 & \lambda_C^{-2} \\ 1 & \lambda_F^{-2} \end{bmatrix}^{-1} = \frac{1}{1.9097 \times 10^{-6}\,\text{nm}^{-2}} \begin{bmatrix} 4.231\,5 \times 10^{-6}\,\text{nm}^{-2} & -2.321\,8 \times 10^{-6}\,\text{nm}^{-2} \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.215\,8 & -1.215\,8 \\ -5.236\,4 \times 10^{5}\,\text{nm}^{2} & 5.236\,4 \times 10^{5}\,\text{nm}^{2} \end{bmatrix}$$

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 2.215\,8 & -1.215\,8 \\ -5.236\,4 \times 10^{5}\,\text{nm}^{2} & 5.236\,4 \times 10^{5}\,\text{nm}^{2} \end{bmatrix} \begin{bmatrix} 1.51418 \\ 1.52225 \end{bmatrix}$$

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 1.504\,4 \\ 4225.8\,\text{nm}^2 \end{bmatrix}$$

at $\lambda_D = 589.59\,\text{nm}$:

$$\boxed{\textit{Crown glass: } n_F = A_0 + \frac{B_0}{\lambda_D^2} = 1.504\,4 + \frac{4225.8\,\text{nm}^2}{(589.59\,\text{nm})^2} = 1.5166\text{, identical to measurement}}$$

For the flint glass:

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} 2.215\,8 & -1.215\,8 \\ -5.236\,4 \times 10^{5}\,\text{nm}^{2} & 5.236\,4 \times 10^{5}\,\text{nm}^{2} \end{bmatrix} \begin{bmatrix} 1.69427 \\ 1.71748 \end{bmatrix} = \begin{bmatrix} 1.666\,1 \\ 12154.\,\text{nm}^2 \end{bmatrix}$$

at $\lambda_D = 589.59\,\text{nm}$:

$$\boxed{\textit{Flint Glass: } n_F = A_0 + \frac{B_0}{\lambda_D^2} = 1.666\,1 + \frac{12154.\,\text{nm}^2}{(589.59\,\text{nm})^2} = 1.701\,1 \text{ vs. } 1.70100}$$

3. (Fowles 2.5) The electric vector of a wave is given by the real expression:

$$\underline{\mathbf{E}}[z,t] = E_0\left[\hat{\underline{\mathbf{x}}}\cos[k_0z-\omega_0t] + \hat{\underline{\mathbf{y}}}\cdot b\cos[k_0z-\omega_0t+\phi_0]\right]$$

Show that this is equivalent to the complex-valued expression:

$$\underline{\mathbf{E}}[z,t] = E_0\left(\hat{\underline{\mathbf{x}}} + \hat{\underline{\mathbf{y}}}\cdot b\exp[+i\phi_0]\right)\exp[+i(k_0z-\omega_0t)]$$

$$\begin{aligned}
\underline{\mathbf{E}}[z,t] &= E_0\left[\hat{\underline{\mathbf{x}}}\cos[k_0z-\omega_0t] + \hat{\underline{\mathbf{y}}}\cdot b\cos[k_0z-\omega_0t+\phi_0]\right] \\
&= \mathrm{Re}\left\{E_0\left[\hat{\underline{\mathbf{x}}}\exp[+i(k_0z-\omega_0t)] + \hat{\underline{\mathbf{y}}}\cdot b\exp[+i(k_0z-\omega_0t+\phi_0)]\right]\right\} \\
&= E_0\cdot\mathrm{Re}\left\{\hat{\underline{\mathbf{x}}}\exp[+i(k_0z-\omega_0t)] + \hat{\underline{\mathbf{y}}}\cdot b\exp[+i(k_0z-\omega_0t)]\cdot\exp[+i\phi_0]\right\} \\
&= E_0\cdot\mathrm{Re}\left\{\left(\hat{\underline{\mathbf{x}}} + \hat{\underline{\mathbf{y}}}\cdot b\cdot\exp[+i\phi_0]\right)\exp[+i(k_0z-\omega_0t)]\right\}
\end{aligned}$$

4. Sketch diagrams to show the type of polarizations in #3 for the following cases:

   (a) $\phi_0 = 0, b = 1$

$$\begin{aligned}
\underline{\mathbf{E}}[z,t] &= E_0\left(\hat{\underline{\mathbf{x}}} + \hat{\underline{\mathbf{y}}}\right)\exp[+i(k_0z-\omega_0t)] \\
&= E_0\cdot\begin{bmatrix}1\\1\end{bmatrix}\cdot\exp[+i(k_0z-\omega_0t)] \\
&\Longrightarrow LP \text{ with amplitude } E_0\cdot\sqrt{1^2+1^2} = \sqrt{2}\cdot E_0 \\
\text{at angle } \theta &= \tan^{-1}\left[\frac{1}{1}\right] = \frac{\pi}{4} = 45^\circ
\end{aligned}$$

   (b) $\phi_0 = 0, b = 2$

$$\begin{aligned}
\underline{\mathbf{E}}[z,t] &= E_0\left(\hat{\underline{\mathbf{x}}} + \hat{\underline{\mathbf{y}}}\cdot 2\right)\exp[+i(k_0z-\omega_0t)] \\
&= E_0\cdot\begin{bmatrix}1\\2\end{bmatrix}\cdot\exp[+i(k_0z-\omega_0t)] \\
&\Longrightarrow LP \text{ with amplitude } E_0\cdot\sqrt{1^2+1^2} = \sqrt{2}\cdot E_0 \\
\text{at angle } \theta &= \tan^{-1}\left[\frac{2}{1}\right] \equiv 1.1071 \text{ radians} \cong 0.3524\pi \text{ radians} \cong 63.44^\circ
\end{aligned}$$

   (c) $\phi_0 = +\frac{\pi}{4}, b = -1$

$$\begin{aligned}
\underline{\mathbf{E}}[z,t] &= E_0\left(\hat{\underline{\mathbf{x}}} - \hat{\underline{\mathbf{y}}}\exp\left[+i\frac{\pi}{4}\right]\right)\exp[+i(k_0z-\omega_0t)] \\
&= E_0\cdot\begin{bmatrix}1\\-1\exp\left[+i\frac{\pi}{4}\right]\end{bmatrix}\cdot\exp[+i(k_0z-\omega_0t)] \\
\underline{\mathcal{E}}_3 &= \begin{bmatrix}1\\-1\exp\left[+i\frac{\pi}{4}\right]\end{bmatrix} = \begin{bmatrix}1\\\exp\left[+i\left(\frac{\pi}{4}-\pi\right)\right]\end{bmatrix} = \begin{bmatrix}1\\\exp\left[-i\frac{3\pi}{4}\right]\end{bmatrix} \\
&\Longrightarrow LHEP \text{ with major axis at angle } \theta = \tan^{-1}\left[\frac{1}{1}\right] = \frac{\pi}{4} = 45^\circ
\end{aligned}$$

5. Write down the Jones vectors for the three cases in the previous problem.

   Did it within the problems

   (a)

$$\underline{\mathcal{E}}_1 = \begin{bmatrix}1\\1\end{bmatrix}$$

   (b)

$$\underline{\mathcal{E}}_2 = \begin{bmatrix}1\\2\end{bmatrix}$$

   (c)

$$\underline{\mathcal{E}}_3 = \begin{bmatrix}1\\-1\exp\left[+i\frac{\pi}{4}\right]\end{bmatrix} = \begin{bmatrix}1\\\exp\left[-i\frac{3\pi}{4}\right]\end{bmatrix}$$

6. For the following three Jones vectors:

$$\underline{\mathcal{E}}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$
$$\underline{\mathcal{E}}_2 = \begin{bmatrix} +i \\ -1 \end{bmatrix}$$
$$\underline{\mathcal{E}}_3 = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix}$$

(a) Determine the type of polarization of each wave;

$$\underline{\mathcal{E}}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = E_0 \begin{bmatrix} \cos[\theta] \\ \sin[\theta] \end{bmatrix}$$
$$\theta = \tan^{-1}\left[\frac{\sqrt{3}}{1}\right] = \frac{\pi}{3}$$
$$E_0 = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$\boxed{\textit{Linearly polarized with amplitude of 2 at angle of } 60^\circ = \frac{\pi}{3}}$

$$\underline{\mathcal{E}}_2 = \begin{bmatrix} +i \\ -1 \end{bmatrix} = +i \begin{bmatrix} 1 \\ \frac{-1}{+i} \end{bmatrix} = +i \begin{bmatrix} 1 \\ +i \end{bmatrix} = \left(\exp\left[+i\frac{\pi}{2}\right]\right) \begin{bmatrix} 1 \\ 1 \cdot \exp\left[+i\frac{\pi}{2}\right] \end{bmatrix}$$

$\boxed{\textit{RHCP with unit amplitude}}$

$$\underline{\mathcal{E}}_3 = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = -(1-i) \begin{bmatrix} 1 \\ \frac{1+i}{1-i} \end{bmatrix} = (1-i) \begin{bmatrix} 1 \\ \frac{(1+i)}{(1-i)}\frac{(1+i)}{(1+i)} \end{bmatrix} = \left(\sqrt{2}\exp\left[-i\frac{\pi}{2}\right]\right) \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\boxed{\textit{RHCP with amplitude } \sqrt{2}}$

(b) Find Jones vectors that are orthogonal to each of the three cases and describe the state of polarization.

Find a vector such that the scalar product is zero, where the definition of the scalar product for vectors with complex-valued components includes a complex conjugate

$$\underline{\mathcal{E}}_1 \bullet \underline{\mathcal{E}}_1^{\perp} = \sum_{n=1}^{2} (\underline{\mathcal{E}}_1)_n \left(\underline{\mathcal{E}}_1^{\perp}\right)_n^* = 0 \Longrightarrow (\underline{\mathcal{E}}_1)_x \left(\underline{\mathcal{E}}_1^{\perp}\right)_x^* + (\underline{\mathcal{E}}_1)_y \left(\underline{\mathcal{E}}_1^{\perp}\right)_y^*$$
$$\underline{\mathcal{E}}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \Longrightarrow \underline{\mathcal{E}}_1^{\perp} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}^* = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$$

check by evaluating scalar product

$$\underline{\mathcal{E}}_1 \bullet \underline{\mathcal{E}}_1^{\perp} = 1 \cdot \sqrt{3}^* + \left(-\sqrt{3} \cdot 1^*\right) = 1 \cdot \sqrt{3} + \left(-\sqrt{3} \cdot 1\right) = 0$$

$$\underline{\mathcal{E}}_2 = \left(\exp\left[+i\frac{\pi}{2}\right]\right) \begin{bmatrix} 1 \\ 1 \cdot \exp\left[+i\frac{\pi}{2}\right] \end{bmatrix} = \left(\exp\left[+i\frac{\pi}{2}\right]\right) \begin{bmatrix} 1 \\ +i \end{bmatrix}$$
$$\underline{\mathcal{E}}_2^{\perp} = \begin{bmatrix} -\left(\exp\left[+i\frac{\pi}{2}\right]\right)^* \\ 1 \end{bmatrix} = \begin{bmatrix} +i \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\underline{\mathcal{E}}_3 = \left(\sqrt{2}\exp\left[-i\frac{\pi}{2}\right]\right) \cdot \begin{bmatrix} 1 \\ +i \end{bmatrix}$$
$$\underline{\mathcal{E}}_3^{\perp} = \begin{bmatrix} +i \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

7. (P^3 15-1) Initially unpolarized light passes in turn through three linear polarizers with transmission axes at 0°, 30°, and 60°, respectively, relative to the horizontal axis. What is the irradiance of the product light expressed as a percentage of the unpolarized light irradiance?

this is Malus' law implemented twice, plus recognizing that the irradiance is the squared magnitude of the amplitude. The first polarizer reduces the irradiance by half and the light is linearly polarized horizontally, so the Jones vector after the first polarizer is:

$$\underline{\mathcal{E}}_1 = \sqrt{\frac{I_0}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The Jones matrix for the second polarizer is:

$$\begin{aligned} \underline{\mathbf{M}}_2 &= \begin{bmatrix} \cos^2\left[\frac{\pi}{6}\right] & \cos\left[\frac{\pi}{6}\right]\sin\left[\frac{\pi}{6}\right] \\ \cos\left[\frac{\pi}{6}\right]\sin\left[\frac{\pi}{6}\right] & \sin^2\left[\frac{\pi}{6}\right] \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \\ (n.b., \det \underline{\mathbf{M}}_2 &= 0) \end{aligned}$$

The angle of the third polarizer is 60° :

$$\underline{\mathbf{M}}_3 = \begin{bmatrix} \cos^2\left[\frac{\pi}{3}\right] & \cos\left[\frac{\pi}{3}\right]\sin\left[\frac{\pi}{3}\right] \\ \cos\left[\frac{\pi}{3}\right]\sin\left[\frac{\pi}{3}\right] & \sin^2\left[\frac{\pi}{3}\right] \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

The output state is the product of the matrices with the input state:

$$\begin{aligned} \underline{\mathcal{E}}_3 &= \underline{\mathbf{M}}_3\underline{\mathbf{M}}_2\underline{\mathcal{E}}_1 = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \sqrt{\frac{I_0}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{3}{8}\sqrt{\frac{3}{2}I_0} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix} \\ I_3 &= \underline{\mathcal{E}}_3 \bullet \underline{\mathcal{E}}_3 = \frac{9}{64} \cdot \frac{3}{2} \cdot I_0 \cdot \left(\frac{1}{3} + 1\right) = \frac{9}{32} I_0 : \boxed{I_3 = \frac{9}{32} I_0 \cong 0.28 I_0} \\ \theta &= \tan^{-1}\left[\sqrt{3}\right] = \frac{\pi}{3} \text{ (as it should!)} \end{aligned}$$

8. (P^3 15-3): Since a sheet of Polaroid is not an ideal polarizer, not all the energy of the **E**-vibrations parallel to the TA are transmitted, nor are all **E**-vibrations perpendicular to the transmission axis are absorbed. Suppose an energy fraction α us transmitted in the first case and a fraction β is transmitted in the second.

   (a) Extend Malus' law by calculating the irradiance transmitted by a pair of such polarizers with angle θ between their transmission axes. Assume initially unpolarized light of irradiance I_0. Show that Malus' law follows in the ideal case.

$$\textit{If two polarizers oriented at angle } \theta \quad : $$
$$\textit{Ideal Malus' Law} \quad : \quad I_1 = I_0 \cos^2\theta$$

We know that half of irradiance of unpolarized light is blocked by an ideal polarizer, so that if a percentage α is passed of the light at angle θ and a percentage β is passed at the orthogonal angle, then some light gets through in all cases.
Assume that the initial polarizer is oriented along x and the second at the angle θ relative to x:

$$\textit{Light passed by x-polarizer oriented along } x \quad : \quad (I_1)_x = \frac{1}{2} I_0 \cdot \alpha$$
$$\textit{Light passed by x-polarizer oriented along } y \quad : \quad (I_1)_y = \frac{1}{2} I_0 \cdot \beta$$

$$\textit{x-axis light passed by polarizer \#2 oriented along } \theta\textit{:}$$
$$(I_{2x})_\theta = (I_1)_x \left(\alpha \cos^2\theta\right) = \frac{1}{2} I_0 \alpha^2 \cos^2\theta$$
$$\textit{y-axis light passed by polarizer \#2 oriented along } \theta\textit{:}$$
$$(I_{2y})_\theta = (I_1)_y \left(\alpha \cos^2\left(\theta + \frac{\pi}{2}\right)\right) = \frac{1}{2} I_0 \alpha\beta \sin^2\theta$$

$$\textit{x-axis light passed by polarizer \#2 in orthogonal direction}:$$
$$(I_{2x})_{\theta+\frac{\pi}{2}} = (I_1)_x \left(\beta \cos^2\left(\theta + \frac{\pi}{2}\right)\right) = \frac{1}{2} \alpha I_0 \beta \sin^2\theta$$
$$\textit{y-axis light passed by polarizer \#2 in orthogonal direction}$$
$$: (I_{2x})_{\theta+\frac{\pi}{2}} = (I_1)_y \left(\beta \cos^2(\theta)\right) = \frac{1}{2} \beta I_0 \left(\beta \cos^2\theta\right)$$

Total light passed:

$$I_2 = \frac{1}{2} I_0 \left(\alpha^2 \cos^2\theta + 2\alpha\beta \sin^2\theta + \beta^2 \cos^2\theta\right)$$

Check limiting behavior of total light passed: set $\alpha = 1$, $\beta = 0$:

$$I_2 = \frac{1}{2} I_0 \left(\alpha 1^2 \cos^2\theta + 2 \cdot 1 \cdot 0 \sin^2\theta + 0^2 \cos^2\theta\right) = \frac{I_0}{2} \cos^2[\theta]$$

identical to Malus' law for ideal polarizer:

$$I_2 = I_0 \cdot \frac{1}{2} \cdot \cos^2[\theta] = \left(\frac{I_0}{2}\right) \cdot \cos^2[\theta]$$

   (b) Let $\alpha = 0.95$ and $\beta = 0.05$ for a given sheet of Polaroid. Compare the irradiance with that of an ideal polarizer when unpolarized light is passed through two such sheets have relative angle between transmission axes of 0°, 30°, 45°, and 90°.

$$\textit{Ideal Malus' Law: } I_2 = I_0 \cdot \frac{1}{2} \cdot \cos^2[\theta]$$
$$\textit{Realistic Malus' Law: } I_2 = \frac{1}{2} I_0 \left(0.95^2 \cos^2\theta + 2 \cdot 0.95 \cdot 0.05 \sin^2\theta + 0.05^2 \cos^2\theta\right)$$
$$= \frac{1}{2} I_0 \left(0.905 \cos^2\theta + 0.095 \sin^2\theta\right)$$

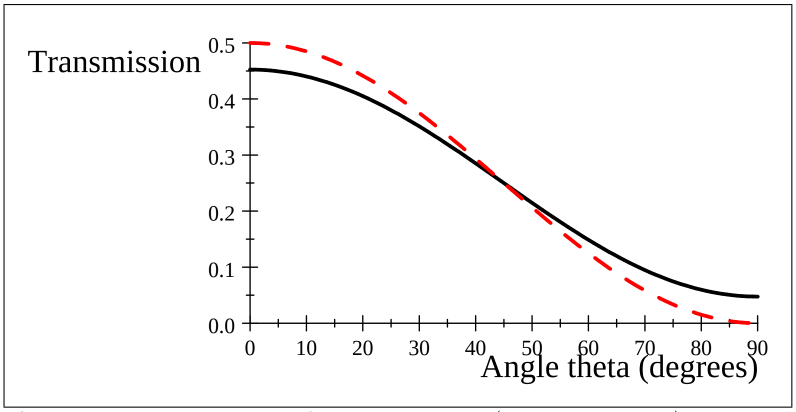
$$\theta = 0 \implies I_2 = \frac{1}{2} I_0 \left(0.905 \cos^2 0 + 0.095 \sin^2 0\right)$$
$$= \frac{0.905}{2} I_0 = 0.4525 I_0$$

$$\theta = \frac{\pi}{6} \implies I_2 = \frac{1}{2} I_0 \left(0.905 \cos^2 \frac{\pi}{6} + 0.095 \sin^2 \frac{\pi}{6}\right)$$
$$= 0.351\,25 I_0$$

$$\theta = \frac{\pi}{4} \implies I_2 = \frac{1}{2} I_0 \left(0.905 \cos^2 \frac{\pi}{4} + 0.095 \sin^2 \frac{\pi}{4}\right)$$
$$= \frac{I_0}{2} \left(\frac{0.905}{2} + \frac{0.095}{2}\right) = I_0 \left(\frac{0.905}{4} + \frac{0.095}{4}\right) = 0.25 I_0$$

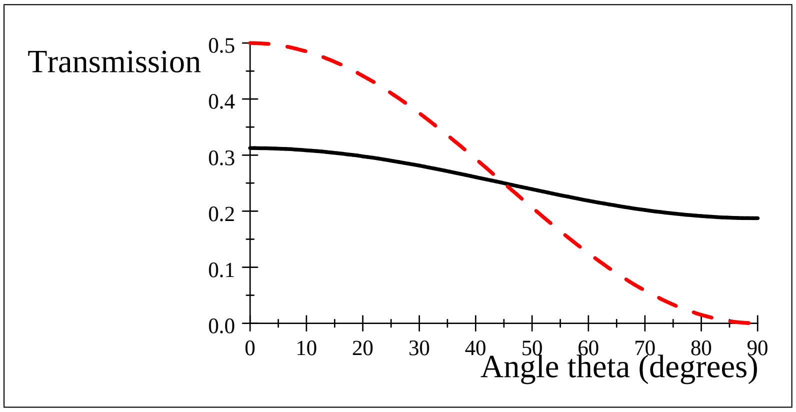
$$\theta = \frac{\pi}{3} \implies I_2 = \frac{1}{2} I_0 \left(0.905 \cos^2 \frac{\pi}{3} + 0.095 \sin^2 \frac{\pi}{3}\right)$$
$$= 0.148\,75 I_0$$

Just for fun, plot the ideal equation and the realisitic equations for these values of α *and* β *with* $I_0 = 1$



Comparison of the realistic expression for Malus' Law (black solid line) to the ideal expression (red dashed line) for $a = 0.95$, $\beta = 0.05$. *Note that the expressions are equal at* $\theta = 45°$.

Try it again for different values of $\alpha = 0.75, \beta = 0.25$:



which shows that the sensitivity of the amount of transmitted light the the angle of the polarizers has become poor.