1051-455-20073 Solution Set #2

1. For the following two harmonic waves:

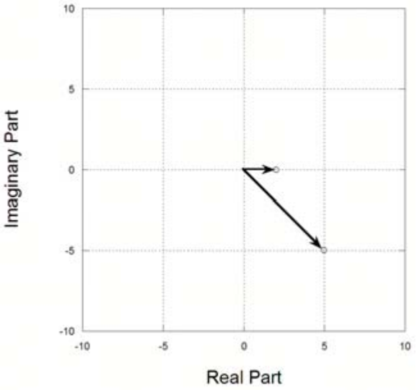
   (a) Show on a phasor diagram:

$$\begin{aligned} f_1[t] &= 2\cos[\omega_0 t] \\ f_2[t] &= 7\cos\left[\omega_0 t - \frac{\pi}{4}\right] \end{aligned}$$

phasor is the complex vector evaluated at $t = 0$*:*

$$\begin{aligned} f_1[t] &= 2\exp[i\cdot 0] = 2 + 0i \\ f_2[t] &= 7\exp\left[-i\frac{\pi}{4}\right] = 7\left(\cos\left[-\frac{\pi}{4}\right] + i\sin\left[-\frac{\pi}{4}\right]\right) \\ &= 7\left(\cos\left[+\frac{\pi}{4}\right] - i\sin\left[\frac{\pi}{4}\right]\right) = \frac{7}{\sqrt{2}}(1-i) \end{aligned}$$

As time increases, the two phasors rotate about the origin at the rate ω_0.



   (b) Find the mathematical expression for the superposition $f_1[t] + f_2[t]$ in the form of a cosine.

$$\begin{aligned} \operatorname{Re}\{f_1[0] + f_2[0]\} &= 2 + \frac{7}{\sqrt{2}} \cong 6.9497 \\ \operatorname{Im}\{f_1[0] + f_2[0]\} &= 0 - \frac{7}{\sqrt{2}} \cong -4.9497 \\ |f_1[0] + f_2[0]| &= \sqrt{\left(2 + \frac{7}{\sqrt{2}}\right)^2 + \left(-\frac{7}{\sqrt{2}}\right)^2} \\ &\cong 8.5322 \\ \Phi\{f_1[0] + f_2[0]\} &= \tan^{-1}\left[\frac{\operatorname{Im}\{f_1[0] + f_2[0]\}}{\operatorname{Re}\{f_1[0] + f_2[0]\}}\right] = \tan^{-1}\left\{\frac{-\frac{7}{\sqrt{2}}}{2 + \frac{7}{\sqrt{2}}}\right\} \\ &\cong -0.6189 \textit{ radians} \cong -0.197\pi \textit{ radians} \cong -35.459^\circ \end{aligned}$$

The sinusoidal expression for this oscillation is:

$$\boxed{f_1[t] + f_2[t] \cong 8.5322 \cdot \cos[\omega_0 t - 0.6189]}$$

2. Two plane waves of the same frequency that vibrate along the $z-$direction are:

$$\begin{aligned} f_1[x,t] &= A_1 \cos\left[2\pi\left(\frac{x}{X_1} - \nu_1 t + \pi\right)\right];\ A_1 = 40\,\text{mm},\ X_1 = 30\,\text{mm},\ \nu_1 = 20\,\text{Hz} \\ f_2[y,t] &= A_2 \cos\left[2\pi\left(\frac{y}{Y_2} - \nu_2 t + \pi\right)\right];\ A_2 = 20\,\text{mm},\ Y_2 = 40\,\text{mm},\ \nu_2 = 20\,\text{Hz} \end{aligned}$$

Evaluate the resultant waveform $f_1[x,t] + f_2[y,t]$ at $[x,y] = [50\,\text{mm}, 20\,\text{mm}]$

full credit if done either way:

(a)

$$\begin{aligned} & f_1[50\,\text{mm},t] + f_2[20\,\text{mm},t] \\ = \;& 40\,\text{mm}\cdot\cos\left[2\pi\left(\frac{50\,\text{mm}}{30\,\text{mm}} - 20\,\text{Hz}\cdot t\right) + \pi\right] + 20\,\text{mm}\cdot\cos\left[2\pi\left(\frac{20\,\text{mm}}{40\,\text{mm}} - 20\,\text{Hz}\cdot t\right) + \pi\right] \\ = \;& 40\,\text{mm}\cdot\left(-\cos\left[2\pi\left(\frac{5}{3} - 20\,\text{Hz}\cdot t\right)\right]\right) + 20\,\text{mm}\cdot\left(-\cos\left[2\pi\left(\frac{1}{2} - 20\,\text{Hz}\cdot t\right)\right]\right) \\ = \;& -40\,\text{mm}\cdot\cos\left[\frac{10\pi}{3} - 40\pi t\right] - 20\,\text{mm}\cdot\cos[\pi - 40\pi t] \\ = \;& -40\,\text{mm}\cdot\cos\left[\frac{10\pi}{3} - 40\pi t\right] + 20\,\text{mm}\cdot\cos[-40\pi t] \\ = \;& -40\,\text{mm}\cdot\cos\left[40\pi t - \frac{10\pi}{3}\right] + 20\,\text{mm}\cdot\cos[40\pi t] \\ = \;& \text{Re}\left\{20\,\text{mm}\cdot\exp[+i40\pi t] - 40\,\text{mm}\cdot\exp\left[+i\left(40\pi t - \frac{10\pi}{3}\right)\right]\right\} \end{aligned}$$

Because the temporal frequencies are the same, we can evaluate the sum at $t = 0$ *by using phasors:*

$$\begin{aligned} & f_1[50\,\text{mm},0] + f_2[20\,\text{mm},0] \\ = \;& 20\,\text{mm}\cdot\exp[+i40\pi\cdot 0] - 40\,\text{mm}\cdot\exp\left[+i\left(40\pi\cdot 0 - \frac{10\pi}{3}\right)\right] \\ = \;& 20\,\text{mm} - 40\,\text{mm}\cdot\exp\left[-i\frac{10\pi}{3}\right] \end{aligned}$$

$$\begin{aligned} \text{Re}\{f_1[50\,\text{mm},0] + f_2[20\,\text{mm},0]\} &= 20\,\text{mm} - 40\,\text{mm}\cdot\cos\left[-\frac{10\pi}{3}\right] = +40\,\text{mm} \\ \text{Im}\{f_1[50\,\text{mm},0] + f_2[20\,\text{mm},0]\} &= -40\,\text{mm}\cdot\sin\left[-\frac{10\pi}{3}\right] = -20\sqrt{3}\,\text{mm} \cong -34.641\,\text{mm} \end{aligned}$$

$$\begin{aligned} |f_1[50\,\text{mm},0] + f_2[20\,\text{mm},0]| &= \sqrt{(+40\,\text{mm})^2 + (-34.641\,\text{mm})^2} \cong 52.915\,\text{mm} \\ \Phi\{f_1[50\,\text{mm},0] + f_2[20\,\text{mm},0]\} &= \tan^{-1}\left[\frac{-34.641}{+40}\right] \\ &\cong -0.71371\ \textit{radians} \cong -0.22718\pi\ \textit{radians}\ \cong -40.892^\circ \end{aligned}$$

(b)

$$\begin{aligned}
& f_1\left[50\,\text{mm}, t\right] + f_2\left[20\,\text{mm}, t\right] \\
= \quad & 40\,\text{mm} \cdot \cos\left[2\pi\left(\frac{50\,\text{mm}}{30\,\text{mm}} - 20\,\text{Hz} \cdot t + \pi\right)\right] + 20\,\text{mm} \cdot \cos\left[2\pi\left(\frac{20\,\text{mm}}{40\,\text{mm}} - 20\,\text{Hz} \cdot t + \pi\right)\right] \\
= \quad & 40\,\text{mm} \cdot -\cos\left[2\pi\left(\frac{5}{3} - 20\,\text{Hz} \cdot t\right)\right] - 20\,\text{mm} \cdot \cos\left[2\pi\left(\frac{1}{2} - 20\,\text{Hz} \cdot t\right) + 2\pi^2\right] \\
= \quad & -40\,\text{mm} \cdot \cos\left[\left(\frac{10\pi}{3} + 2\pi^2\right) - 40\pi t\right] - 20\,\text{mm} \cdot \cos\left[\left(\pi + 2\pi^2\right) - 40\pi t\right] \\
= \quad & -40\,\text{mm} \cdot \cos\left[40\pi t - \left(\frac{10\pi}{3} + 2\pi^2\right)\right] - 20\,\text{mm} \cdot \cos\left[40\pi t - \left(\pi + 2\pi^2\right)\right] \\
= \quad & \text{Re}\left\{-40\,\text{mm} \cdot \exp\left[+i\left(40\pi t - \left(\frac{10\pi}{3} + 2\pi^2\right)\right)\right] - 20\,\text{mm} \cdot \exp\left[+i\left(40\pi t - \left(\pi + 2\pi^2\right)\right)\right]\right\}
\end{aligned}$$

Because the temporal frequencies are the same, we can evaluate the sum at $t = 0$ *by using phasors:*

$$\begin{aligned}
\text{Re}\left\{f_1\left[50\,\text{mm}, 0\right] + f_2\left[20\,\text{mm}, 0\right]\right\} &= -40\,\text{mm} \cdot \cos\left[\frac{10\pi}{3} + 2\pi^2\right] - 20\,\text{mm} \cdot \cos\left[\pi + 2\pi^2\right] \\
& -40\,\text{mm} \cdot \cos\left[\frac{10\pi}{3} + 2\pi^2\right] - 20\,\text{mm} \cdot \cos\left[\pi + 2\pi^2\right] \\
&\cong -1.7237\,\text{mm} \\
\text{Re}\left\{f_1\left[50\,\text{mm}, 0\right] + f_2\left[20\,\text{mm}, 0\right]\right\} &= -40\,\text{mm} \cdot \sin\left[\frac{10\pi}{3} + 2\pi^2\right] - 20\,\text{mm} \cdot \sin\left[\pi + 2\pi^2\right] \\
&\cong +52.887\,\text{mm}
\end{aligned}$$

$$\begin{aligned}
\left|f_1\left[50\,\text{mm}, 0\right] + f_2\left[20\,\text{mm}, 0\right]\right| &= \sqrt{\left(-1.7237\,\text{mm}\right)^2 + \left(+52.887\,\text{mm}\right)^2} \cong 52.915\,\text{mm} \\
\Phi\left[f_1\left[50\,\text{mm}, 0\right] + f_2\left[20\,\text{mm}, 0\right]\right] &= \tan^{-1}\left\{\frac{+52.887}{-1.7237}\right\} \cong -1.5382 \textit{ radians} \cong -88.134^\circ
\end{aligned}$$

3. Consider the superposition of two sinusoidal traveling waves:

$$\begin{aligned}
f_1[z,t] &= A_1 \cos[k_1 z - \omega_1 t], \\
A_1 &= 101\,\mathrm{mm},\ \nu_1 = 100\,\mathrm{Hz},\ \mathrm{v}_1 = 250\frac{\mathrm{m}}{\mathrm{s}} \\
f_2[z,t] &= A_2 \cos[k_2 z - \omega_2 t], \\
A_2 &= 99\,\mathrm{mm},\ \nu_2 = 150\,\mathrm{Hz},\ \mathrm{v}_2 = 500\frac{\mathrm{m}}{\mathrm{s}}
\end{aligned}$$

(a) Find an expression for the resulting wave in terms of the average wave, the modulation wave, plus any remaining amplitude.

We have an expression for the sum of two waves with different frequences bu the same magnitude:

$$\cos[A] + \cos[B] = 2\cos\left[\frac{A+B}{2}\right] \cdot \cos\left[\frac{A-B}{2}\right]$$

so we have:

$$\begin{aligned}
& A_1 \cos[k_1 z - \omega_1 t] + A_2 \cos[k_2 z - \omega_2 t] \\
= {} & [A_2 + (A_1 - A_2)] \cos[k_1 z - \omega_1 t] + A_2 \cos[k_2 z - \omega_2 t] \\
= {} & A_2 (\cos[k_1 z - \omega_1 t] + \cos[k_2 z - \omega_2 t]) + (A_1 - A_2) \cos[k_1 z - \omega_1 t] \\
= {} & 2A_2 \cos\left[\left(\frac{k_1 + k_2}{2}\right) z - \left(\frac{\omega_1 + \omega_2}{2}\right) t\right] \cdot \cos\left[\left(\frac{k_1 - k_2}{2}\right) z - \left(\frac{\omega_1 - \omega_2}{2}\right) t\right] \\
& + (A_1 - A_2) \cos[k_1 z - \omega_1 t]
\end{aligned}$$

which is the sum of a traveling wave and the product of the average and modulation waves

(b) Calculate the wavelengths of the average and modulation waves.

In this case:

$$\begin{aligned}
\lambda_1 \nu_1 &= \mathrm{v}_1 \implies \lambda_1 = \frac{\mathrm{v}_1}{\nu_1} = \frac{250\frac{\mathrm{m}}{\mathrm{s}}}{100\,\mathrm{Hz}} = \frac{5}{2}\,\mathrm{m} \\
\lambda_2 \nu_2 &= \mathrm{v}_2 \implies \lambda_2 = \frac{\mathrm{v}_2}{\nu_2} = \frac{500\frac{\mathrm{m}}{\mathrm{s}}}{150\,\mathrm{Hz}} = \frac{10}{3}\,\mathrm{m} \\
&\implies \lambda_1 > \lambda_2 \text{ and } \mathrm{v}_1 < \mathrm{v}_2
\end{aligned}$$

$$\begin{aligned}
k_{avg} &= \frac{2\pi}{\lambda_{avg}} = \frac{k_1 + k_2}{2} = 2\pi \frac{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}{2} \\
&\implies \lambda_{avg} = \left(\frac{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}{2}\right)^{-1} = \left(\frac{\frac{2}{5\,\mathrm{m}} + \frac{3}{10\,\mathrm{m}}}{2}\right)^{-1} = \boxed{\frac{20}{7}\,\mathrm{m} = \lambda_{avg}}
\end{aligned}$$

$$\begin{aligned}
k_{\mathrm{mod}} &= \frac{2\pi}{\lambda_{\mathrm{mod}}} = \frac{k_1 - k_2}{2} = 2\pi \frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}{2} \\
&\implies \lambda_{\mathrm{mod}} = \left(\frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}{2}\right)^{-1} = \left(\frac{\frac{2}{5\,\mathrm{m}} - \frac{3}{10\,\mathrm{m}}}{2}\right)^{-1} = \boxed{20\,\mathrm{m} = \lambda_{\mathrm{mod}}}
\end{aligned}$$

n.b., $\lambda_{\mathrm{mod}} > \lambda_{avg}$

(c) Find the velocities of the average and modulation waves.

$$\begin{aligned}\omega_{avg} &= \frac{\omega_1+\omega_2}{2} = 2\pi\left(\frac{\nu_1+\nu_2}{2}\right) = 2\pi\left(\frac{100\,\text{Hz}+150\,\text{Hz}}{2}\right) = 2\pi\cdot 125 \\ \omega_{\text{mod}} &= \frac{\omega_1-\omega_2}{2} = 2\pi\left(\frac{\nu_1-\nu_2}{2}\right) = 2\pi\left(\frac{100\,\text{Hz}-150\,\text{Hz}}{2}\right) = -2\pi\cdot 25\end{aligned}$$

$$\begin{aligned}\text{v}_{avg} &= \frac{\omega_{avg}}{k_{avg}} = \lambda_{avg}\cdot\nu_{avg} = \frac{20}{7}\,\text{m}\cdot 125\,\text{Hz} = \boxed{\frac{2500}{7}\frac{\text{m}}{\text{s}} = \text{v}_{avg}} \\ \text{v}_{\text{mod}} &= \frac{\omega_{\text{mod}}}{k_{\text{mod}}} = \lambda_{\text{mod}}\cdot\nu_{\text{mod}} = 20\,\text{m}\cdot -25\,\text{Hz} = \boxed{500\frac{\text{m}}{\text{s}} = \text{v}_{\text{mod}}}\end{aligned}$$

so the modulation wave travels in the opposite direction

(d) Does this system exhibit normal or anomalous dispersion?

$$\boxed{\lambda_1 > \lambda_2 \textit{ and } \text{v}_1 < \text{v}_2 \implies \textit{longer wave travels slower} \implies \textit{anomalous dispersion}}$$

4. The phase velocity of waves in some medium is proportional to $\omega^{+\frac{1}{2}}$. Find an expression for the modulation velocity and determine whether the waves exhibit normal or anomalous dispersion.

$$\begin{aligned}
\mathrm{v}_\phi &= \frac{\omega}{k} = \alpha\omega^{+\frac{1}{2}}, \textit{ where } \alpha \textit{ is some constant} \\
&\Longrightarrow k = \frac{\omega}{\alpha\omega^{+\frac{1}{2}}} \Longrightarrow \alpha k = \omega^{+\frac{1}{2}} \Longrightarrow \boxed{\omega = (\alpha k)^2} \\
&\quad \textit{this is the dispersion relation } \omega\,[k] \\
\mathrm{v}_{\mathrm{mod}} &= \frac{d\omega}{dk} = \frac{d}{dk}\left[(\alpha k)^2\right] = 2\alpha^2 k = 2\alpha^2\left(\frac{\omega^{+\frac{1}{2}}}{\alpha}\right) = 2\alpha\omega^{+\frac{1}{2}} \\
\mathrm{v}_{\mathrm{mod}} &= 2\alpha\omega^{+\frac{1}{2}} = 2\mathrm{v}_{avg} \Longrightarrow \boxed{\textit{anomalous dispersion}}
\end{aligned}$$

5. Plot and write the equation of the superposition of the following harmonic waves:

$$\begin{aligned}
E_1 &= \sin\left[\frac{\pi}{18} - \omega t\right] \\
E_2 &= 3\cos\left[\frac{5\pi}{9} - \omega t\right] \\
E_3 &= 2\sin\left[\frac{\pi}{6} - \omega t\right]
\end{aligned}$$

where the period of each is 2 s.

$$\begin{aligned}
E_1 &= \sin\left[\frac{\pi}{18} - \omega t\right] = \cos\left[\left(\frac{\pi}{18} - \omega t\right) - \frac{\pi}{2}\right] \\
&= \cos\left[\left(\frac{\pi}{18} - \frac{\pi}{2}\right) - \omega t\right] = \cos\left[\omega t - \left(-\frac{8\pi}{18}\right)\right] = \cos\left[\omega t + \frac{4\pi}{9}\right]
\end{aligned}$$

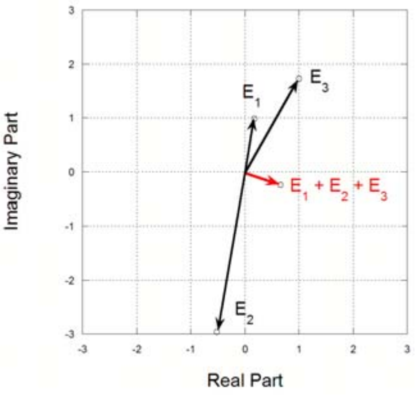
$$\begin{aligned}
E_2 &= 3\cos\left[\frac{5\pi}{9} - \omega t\right] = 3\cos\left[\omega t - \frac{5\pi}{9}\right] \\
E_3 &= 2\sin\left[\frac{\pi}{6} - \omega t\right] = 2\cos\left[\left(\frac{\pi}{6} - \omega t\right) - \frac{\pi}{2}\right] \\
&= 2\cos\left[-\frac{\pi}{3} - \omega t\right] = 2\cos\left[\omega t + \frac{\pi}{3}\right]
\end{aligned}$$

since all three components have the same frequency $\omega = 2\pi\nu = \frac{2\pi}{2\,\text{sec}} = \pi\frac{radians}{\text{sec}}$*, the sum has the same frequency. Evaluate the amplitude for* $t = 0$*:*

$$E_1[t=0] + E_2[t=0] + E_3[t=0] = 1 \cdot \exp\left[+i \cdot \frac{4\pi}{9}\right] + 3 \cdot \exp\left[-i \cdot \frac{5\pi}{9}\right] + 2 \cdot \exp\left[+i \cdot \frac{\pi}{3}\right]$$

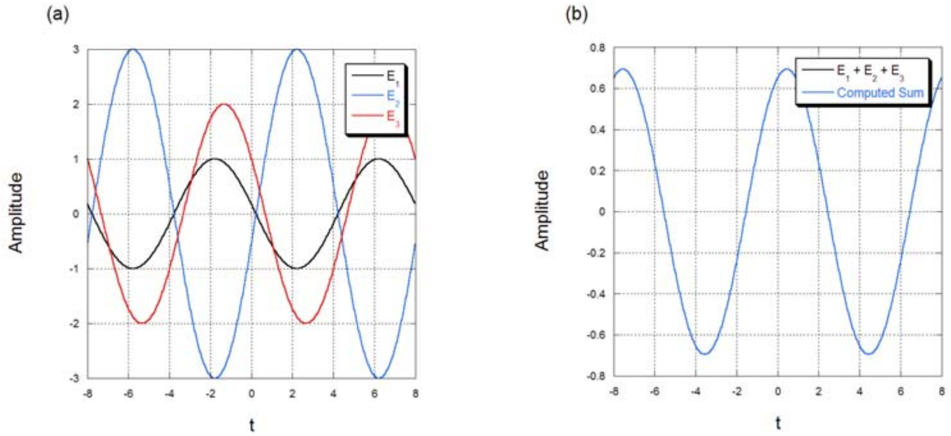
$$\begin{aligned}
\text{Re}\{E_1[t=0] + E_2[t=0] + E_3[t=0]\} &= 1 \cdot \cos\left[+\frac{4\pi}{9}\right] + 3 \cdot \cos\left[-\frac{5\pi}{9}\right] + 2 \cdot \cos\left[+\frac{\pi}{3}\right] \\
&= 1 - 2 \cdot \cos\left[\frac{4}{9}\pi\right] \cong 0.6527
\end{aligned}$$

$$\begin{aligned}
\text{Im}\{E_1[t=0] + E_2[t=0] + E_3[t=0]\} &= 1 \cdot \sin\left[+\frac{4\pi}{9}\right] + 3 \cdot \sin\left[-\frac{5\pi}{9}\right] + 2 \cdot \sin\left[+\frac{\pi}{3}\right] \\
&= \sqrt{3} - 2 \cdot \sin\left[\frac{4}{9}\pi\right] \cong -0.237\,56
\end{aligned}$$



$$\begin{aligned}
|E_1[t=0]+E_2[t=0]+E_3[t=0]| &= \sqrt{\left(1-2\cdot\cos\left[\frac{4}{9}\pi\right]\right)^2+\left(\sqrt{3}-2\cdot\sin\left[\frac{4}{9}\pi\right]\right)^2} \\
&\cong 0.694\,59 \\
\Phi\left\{E_1[t=0]+E_2[t=0]+E_3[t=0]\right\} &= \tan^{-1}\left[\frac{\sqrt{3}-2\cdot\sin\left[\frac{4}{9}\pi\right]}{1-2\cdot\cos\left[\frac{4}{9}\pi\right]}\right] = -\frac{\pi}{9}\ \textit{radians} = -20^\circ
\end{aligned}$$

$$\boxed{E_1+E_2+E_3 \cong 0.694\,59\cdot\cos\left[\omega t-\frac{\pi}{9}\right] \cong 0.695\cdot\cos\left[\omega t-0.349\right]}$$



(a) the three component sinusoids; (b) the sum using both the sum of the three functions in (a) and the computation – the two results are identical

6. A laser emits a monochromatic beam of wavelength λ_0, which is reflected normally from a plane mirror that recedes from the light source at velocity v.

   (a) Determine the beat frequency between the incident and reflected light.

   First try it without for $v=0$*; the reflected light differs from the incident light only in the direction (assuming no losses or phase changes in the mirror):*

$$\begin{aligned} f_1[z,t] &= A_0\cos[k_1 z-\omega_1 t] \\ f_2[z,t] &= A_0\cos[k_1 z+\omega_1 t] \end{aligned}$$

The sum creates standing waves:

$$\begin{aligned} f_1[z,t]+f_2[z,t] &= A_0\cos[k_1 z-\omega_1 t]+A_0\cos[k_1 z+\omega_1 t] \\ &= 2A_0\cos[k_1 z]\cos[\omega_1 t] \end{aligned}$$

Note that if the mirror moves with velocity v, the frequency and the wavelength of the reflected light change due to the Doppler effect. From Eq.(4-44) n Pedrotti(s):

$$\frac{\lambda_{after}}{\lambda_{before}}=\sqrt{\frac{1-\frac{\mathrm{v}}{c}}{1+\frac{\mathrm{v}}{c}}}$$

In the limit v<< c, then Eq.(4-45) applies:

$$\begin{aligned} \frac{\lambda_{after}}{\lambda_{before}} &\equiv \frac{\lambda_0'}{\lambda_0}=\sqrt{\frac{1-\frac{\mathrm{v}}{c}}{1+\frac{\mathrm{v}}{c}}}=\sqrt{\left(1-\frac{\mathrm{v}}{c}\right)\left(1+\frac{\mathrm{v}}{c}\right)^{-1}} \\ &= \sqrt{1-\frac{\mathrm{v}}{c}}\cdot\sqrt{\left(1+\frac{\mathrm{v}}{c}\right)^{-1}} \\ &\cong \sqrt{1-\frac{\mathrm{v}}{c}}\cdot\sqrt{1+(-1)\frac{\mathrm{v}}{c}} \\ &= \sqrt{1-\frac{\mathrm{v}}{c}}\cdot\sqrt{1-\frac{\mathrm{v}}{c}} \\ &= 1-\frac{\mathrm{v}}{c} \\ \lambda_0' &= \left(1-\frac{\mathrm{v}}{c}\right)\lambda_0 \end{aligned}$$

so the wavelength of the reflected beam is longer if v recedes from the source ($v<0$*, red shift) and shorter if it approaches the source (*$v>0$*, blue shift). In our case, the reflected beam exhibits twice the Doppler shift because it has to change direction, so the corresponding wavevectors are:*

$$\begin{aligned} k_1 &= \frac{2\pi}{\lambda_0}\ \textit{unchanged} \\ k_1' &= \frac{2\pi}{\lambda_0'}=\frac{2\pi}{\left(1-\frac{2\mathrm{v}}{c}\right)\lambda_0}=k_1\cdot\frac{1}{1-\frac{2\mathrm{v}}{c}}=k_1\cdot\left(1-\frac{2\mathrm{v}}{c}\right)^{-1}\cong k_1\left(1+\frac{2\mathrm{v}}{c}\right) \end{aligned}$$

where the identity has been used:

$$\frac{1}{1-\alpha}=\sum_{n=0}^{\infty}\alpha^n=1-\alpha+\alpha^2-\cdots\cong 1-\alpha\ \textit{if}\ |\alpha|^2<<|\alpha|$$

The corresponding oscillation frequencies are easy to find because the velocity of light is unchanged:

$$\lambda_0\cdot\nu_0=\lambda_0'\cdot\nu_0'\implies\frac{\lambda_0'}{\lambda_0}=\frac{\nu_0}{\nu_0'}=1-\frac{2\mathrm{v}}{c}$$

:

$$\begin{aligned} &\nu_1 \textit{ is unchanged} \\ \nu_1' &= \frac{\nu_1}{1-\frac{2\mathrm{v}}{c}} \cong \left(1+\frac{2\mathrm{v}}{c}\right)\cdot\nu_1 \end{aligned}$$

So if v is positive, the wavelength decreases and the frequency increases.
The sum of the two waves is:

$$\begin{aligned} f_1[z,t]+f_2[z,t] &= A_0\cos[k_1 z-\omega_1 t]+A_0\cos[k_1' z+\omega_1' t] \\ &= 2A_0\cos[k_{avg}z-\omega_{avg}t]\cdot\cos[k_{\mathrm{mod}}z-\omega_{\mathrm{mod}}t] \end{aligned}$$

where:

$$\begin{aligned} k_{avg} &= \frac{k_1+k_1'}{2}=\frac{1}{2}\left(k_1+k_1\left(1+\frac{2\mathrm{v}}{c}\right)\right)=\frac{1}{2}\left(2k_1+k_1\frac{2\mathrm{v}}{c}\right)=k_1+\frac{k_1\mathrm{v}}{c} \\ k_{\mathrm{mod}} &= \frac{k_1-k_1'}{2}=\frac{1}{2}\left(k_1-k_1\left(1+\frac{2\mathrm{v}}{c}\right)\right)=-\frac{k_1\mathrm{v}}{c} \\ \omega_{avg} &= \frac{\omega_1+\omega_1'}{2}=\frac{1}{2}\left(\omega_1+\left(1+\frac{2\mathrm{v}}{c}\right)\cdot\omega_1\right)=\omega_1+\frac{\mathrm{v}\omega_1}{c} \\ \omega_{\mathrm{mod}} &= \frac{\omega_1-\omega_1'}{2}=\frac{1}{2}\left(\omega_1-\left(1+2\frac{\mathrm{v}}{c}\right)\cdot\omega_1\right)=-\frac{\mathrm{v}\omega_1}{c} \\ \nu_{\mathrm{mod}} &= -\frac{\mathrm{v}\nu_1}{c} \end{aligned}$$

$$\begin{aligned} f_1[z,t]+f_2[z,t] &= A_0\cos[k_1 z-\omega_1 t]+A_0\cos[k_1' z+\omega_1' t] \\ &= 2A_0\cos\left[\left(k_1+\frac{k_1\mathrm{v}}{c}\right)z-\left(\omega_1+\frac{\mathrm{v}\omega_1}{c}\right)t\right]\cdot\cos\left[-\frac{k_1\mathrm{v}}{c}z--\frac{\mathrm{v}\omega_1}{c}t\right] \\ &= 2A_0\cos\left[\left(k_1+\frac{k_1\mathrm{v}}{c}\right)z-\left(\omega_1+\frac{\mathrm{v}\omega_1}{c}\right)t\right]\cdot\cos\left[-\frac{\mathrm{v}}{c}(k_1 z+\omega_1 t)\right] \end{aligned}$$

so the beat frequency in the equation is:

$$|\nu_{\mathrm{mod}}|=\frac{|\omega_{\mathrm{mod}}|}{2\pi}=\frac{1}{2\pi}\frac{\mathrm{v}\omega_1}{c}=\frac{\nu_1}{c}\cdot\mathrm{v}$$

but we actually see the squared magnitude, which oscillates twice as fast (Pedrotti Eq.(5-34))

$$|\nu_{\mathrm{mod}}|=2\cdot\frac{\nu_1}{c}\cdot\mathrm{v}$$

(b) Determine the beat frequency between the incident and reflected light if the light is incident on the plane mirror at angle θ.

Here the reflected beam travels at angle 2θ, *so the* $k-$*vectors are different before and after the reflection. Again consider the case* $v=0$:

$$\begin{aligned}
\underline{\mathbf{k}}_1 &= \begin{bmatrix} 0 \\ 0 \\ \frac{2\pi}{\lambda} \end{bmatrix} = \frac{2\pi}{\lambda}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\underline{\mathbf{k}}_1' &= \begin{bmatrix} \frac{2\pi}{\lambda}\sin[\theta] \\ 0 \\ \frac{2\pi}{\lambda}\cos[\theta] \end{bmatrix} = \frac{2\pi}{\lambda}\begin{bmatrix} \sin[\theta] \\ 0 \\ \cos[\theta] \end{bmatrix} \\
\underline{\mathbf{k}}_{avg} &= \frac{\underline{\mathbf{k}}_1 + \underline{\mathbf{k}}_1'}{2} = \frac{2\pi}{\lambda}\begin{bmatrix} \frac{\sin[\theta]}{2} \\ 0 \\ \frac{1+\cos[\theta]}{2} \end{bmatrix} \\
\underline{\mathbf{k}}_{\text{mod}} &= \frac{\underline{\mathbf{k}}_1 - \underline{\mathbf{k}}_1'}{2} = \frac{2\pi}{\lambda}\begin{bmatrix} -\frac{\sin[\theta]}{2} \\ 0 \\ \frac{1-\cos[\theta]}{2} \end{bmatrix}
\end{aligned}$$

The sum of the two waves is:

$$f_1[x,y,z,t]+f_2[x,y,z,t] = 2\cos\left[\frac{2\pi}{\lambda}\left(\frac{\sin[\theta]}{2}x + \frac{1+\cos[\theta]}{2}z\right) - \omega t\right]\cdot\cos\left[\frac{2\pi}{\lambda}\left(-\frac{\sin[\theta]}{2}x + \frac{1-\cos[\theta]}{2}z\right)\right]$$

so the beat frequency is zero (standing waves).

If we add the movement, we get the same wavelength shifts, but now they depend on the angle:

$$\begin{aligned}
\mathrm{v}_z &= \mathrm{v}\cdot\cos[2\theta] \\
\mathrm{v}_x &= \mathrm{v}\cdot\sin[2\theta]
\end{aligned}$$

$$\nu_{\text{mod}} = 2\frac{\nu_1}{c}\cdot\mathrm{v}\cdot\cos[2\theta]$$