1. Determine which of the following describe traveling waves. Where appropriate, draw the profile and find the speed and direction of motion.

(a) $\psi[y,t] = \exp[-(a^2y^2 + b^2t^2 - 2abty)]$

The argument of a traveling wave must have the form $\Phi[y,t] = k_0 z - \omega_0 t$ where $k_0$ and $\omega_0$ are constants. In this case

$$f[y,t] = \exp[-(a^2y^2 + b^2t^2 - 2abty)]$$
$$= \exp[-(ay - bt)^2] = \exp[-(\Phi[y,t])^2]$$

The argument $u = ay - bt$ does have the proper form for the function $f[u] = \exp[-u^2]$, so this is a traveling wave. The argument remains constant for larger $y$ if $t$ increases, so the form moves to the right towards $+\infty$. Examples are plotted below for different times $t_0$ and $t_1 > t_0$ for $a = b = 1$
(b) $\psi [z, t] = A \sin (az^2 - bt^2)$

we can check this one by applying it to the wave equation $\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$.

$$\Rightarrow v^2 = \left( \frac{\partial^2 \psi}{\partial t^2} \right) \left( \frac{\partial^2 \psi}{\partial z^2} \right)^{-1}$$

$$\frac{\partial \psi}{\partial z} = A \frac{\partial}{\partial z} \sin (az^2 - bt^2) = 2Aaz \cos (bt^2 - az^2)$$

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial}{\partial z} (2Aaz \cos (bt^2 - az^2)) = 2Aa \cos (bt^2 - az^2) + 4Aa^2 z^2 \sin (bt^2 - az^2)$$

$$\frac{\partial \psi}{\partial t} = A \frac{\partial}{\partial t} \sin (az^2 - bt^2) = -2Abt \cos (bt^2 - az^2)$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} (-2Abt \cos (bt^2 - az^2)) = 4Ab^2 t^2 \sin (bt^2 - az^2) - 2Ab \cos (bt^2 - az^2)$$

$$\frac{\left( \frac{\partial^2 \psi}{\partial z^2} \right) \left( \frac{\partial^2 \psi}{\partial t^2} \right)}{4Aa^2 z^2 \sin (bt^2 - az^2) + 2Aa \cos (bt^2 - az^2)}$$

which does not reduce to a velocity, so NOT a traveling wave.

(c) $\psi [x, t] = A \sin \left( 2\pi \left( \frac{x}{a} + \frac{t}{b} \right)^2 \right)$

This has the proper form where the velocity is $v = \lambda_0 \nu_0 = \frac{a}{b}$. The argument shows that $x$ must decrease as $t$ increases, so the function moves to the left towards $x = -\infty$. The function is shown below for $A = 1$ unit, $a = 1$ unit, and $b = 1$ sec for $t_0 = 0$ and $t_1 = 1$:

$f [y, t = 0]$ (black) and $f [y, t = 1]$ (red) for $A = a = b = 1$ unit

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(d) $\psi [x, t] = A \cos^2 [2\pi (t - x)]$

Again has the correct form so it is a travelling wave. As $t$ increases, so must $x$ so this wave moves to the right towards $x = +\infty$. Two plots at different times are shown below for $A = 1$ evaluated at $t = 0$ (black) and $t = 0.1$ (red).

[f(y,t=0) (black) and f(y,t=0.1) (red) for A = 1 unit]
2. The figure shows the profile of a transverse wave on a string traveling in the positive $z$-direction at a speed of $\frac{1 \text{ m}}{s}$.

(a) Determine its wavelength.

*the distance between adjacent maxima or minima is $Z_0 = 300 \text{ mm}$*

(b) Notice that as the wave passes any fixed point on the $z$-axis, the string at that location oscillates in time. Draw a graph of $\psi [t]$ showing how a point on the rope at $z = 0$ oscillates.

*since the velocity is $1 \frac{\text{m}}{s}$, then the function moves past the viewer at one location through $3 \frac{1}{3}$ cycles in one second. The graph of the temporal behavior looks just like that of the spatial behavior except the horizontal axis becomes time in seconds and the unit of $z = 300 \text{ mm}$ becomes $t = \frac{3}{10} \text{ sec}$.*

(c) What is the temporal frequency of the wave?

*As just shown, a wave with period $\lambda_0 = 300 \text{ mm}$ that moves at $v = \frac{1 \text{ m}}{s}$ has one cycle go past in $\frac{3}{10} \text{ sec}$, which means that the frequency is $\frac{10}{3} \text{ Hz}$.*
3. A particle executing simple harmonic motion given by

\[ y[t] = 4 \sin \left[ 2\pi \frac{t}{6} + \phi_0 \right] \]

is displaced by +1 unit when \( t = 0 \). Find:

(a) the phase angle \( \Phi[t = 0] \equiv \phi_0 \)

\[ y[0] = +1 = 4 \sin \left[ 2\pi \frac{0}{6} + \phi_0 \right] = 4 \sin \left[ \phi_0 \right] \]

\[ \Rightarrow \sin \left[ \phi_0 \right] = + \frac{1}{4} \Rightarrow \phi_0 = \sin^{-1} \left[ \frac{1}{4} \right] \]

\[ \cong 0.252 \text{ radians} \cong 14.4^\circ \cong 14^\circ24' \]

\[ \Rightarrow y[t] = 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] \cong 4 \sin \left[ 2\pi \frac{t}{6} + 0.252 \right] \]

(b) the difference in phase between any two positions of the particle separated in time by 2 sec;

\[ t_1 \& t_2 \text{ are any two times such that } t_2 - t_1 = 2 \text{ sec} \]

\[ y[t_2] = 4 \sin \left[ 2\pi \frac{t_2}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] \]

\[ y[t_1] = 4 \sin \left[ 2\pi \frac{t_1}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] \]

\[ \Delta \Phi = \Phi[t_2] - \Phi[t_1] = 2\pi \frac{t_2 - t_1}{6} = 2\pi \frac{2 \text{ sec}}{6 \text{ sec}} = \frac{2\pi}{3} \text{ radians} = 120^\circ \]

(c) the phase angle corresponding to a displacement of +2;

if \( y = +2 \text{ units} \), then the initial equation is:

\[ +2 = 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] = 4 \sin \left[ \Phi[t] \right] \]

\[ \Rightarrow \sin \left[ \Phi[t] \right] = \frac{2}{4} \Rightarrow \Phi[t] = \frac{\pi}{6} \cong 30^\circ \]

(d) the time necessary to reach a displacement of +3 units from the initial position.

if \( y = +3 \text{ units} \), then the initial equation is:

\[ +3 = 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] = 4 \sin \left[ \Phi[t] \right] \]

\[ \Rightarrow \sin \left[ \Phi[t] \right] = \frac{3}{4} \Rightarrow \Phi[t] = \sin^{-1} \left[ \frac{3}{4} \right] \cong 48^\circ40' \]

\[ \Rightarrow 2\pi \frac{t}{6} + 14^\circ24' \cong 48^\circ40' \]

\[ \Rightarrow 2\pi \frac{t}{6} \cong 34^\circ16' \cong 0.598 \Rightarrow t \cong \frac{6}{2\pi} \cdot 0.598 \cong 0.57 \text{ sec} \]
4. A wave vibrates according to the equation

\[ y(z, t) = \frac{1}{2} \sin \left( \frac{\pi z}{3} \right) \cdot \cos (40\pi t) \]

where \( y \) and \( z \) are expressed in mm and \( t \) in sec.

(a) What are the amplitudes and the velocities of the component waves that give rise to this vibration?

This is a standing wave that may be created by adding two waves with the same amplitude, wavelength, and frequency headed in opposite directions via the relationship:

\[
A_0 \cos [\phi_1 (z, t)] + A_0 \cos [\phi_2 (z, t)] = 2A_0 \cos \left[ \frac{\phi_1 (z, t) + \phi_2 (z, t)}{2} \right] \cdot \cos \left[ \frac{\phi_1 (z, t) - \phi_2 (z, t)}{2} \right]
\]

\[
\Rightarrow A_0 \cos [k_0 z - \omega_0 t] + A_0 \cos [k_0 z + \omega_0 t] = 2A_0 \cos [k_0 z] \cdot \cos [\omega_0 t]
\]

so

\[
2A_0 = \frac{1}{2} \text{mm} \quad \Rightarrow \quad A_0 = \frac{1}{4} \text{mm}
\]

\[
k_0 = \frac{\pi}{3} = \frac{2\pi}{6} \quad \Rightarrow \quad \lambda_0 = 6 \text{mm}
\]

\[
\omega_0 = 40\pi \quad \Rightarrow \quad \nu_0 = 20 \text{Hz} \quad \Rightarrow \quad T_0 = \frac{1}{20} \text{sec}
\]

\[
v_\phi = \frac{\omega_0}{k_0} = \lambda_0 \nu_0 = 6 \text{mm} \cdot 20 \text{Hz} = \boxed{120 \text{ mm/sec}}
\]

(b) What is the distance between the nodes?

\[
\sin \left( \frac{\pi z}{3} \right) = \sin \left( \frac{2\pi z}{6} \right) \quad \Rightarrow \quad \text{2 nodes in 6 mm} \quad \Rightarrow \quad \boxed{3 \text{ mm}}
\]

(c) What is the velocity of a particle at the position \( z = 1.5 \text{ mm} \) when \( t = \frac{9}{8} \text{ sec} \)?

\[
y \left[ z = 1.5 \text{ mm}, t = \frac{9}{8} \text{ sec} \right] = \frac{1}{2} \sin \left[ \frac{\pi \cdot 1.5}{3} \right] \cdot \cos \left[ 40\pi \cdot \frac{9}{8} \right]
\]

\[
= \frac{1}{2} \sin \left[ \frac{\pi}{2} \right] \cdot \cos [35\pi]
\]

\[
= \frac{1}{2} \cdot 1 \cdot \cos [\pi] = -\frac{1}{2}
\]

This is an extremum, so \boxed{\text{velocity} = 0 \text{ mm/sec}}
5. By finding appropriate relations for $k \cdot \mathbf{r}$, write equations describing a sinusoidal plane wave in three directions in terms of wavelength and velocity for the three cases:

(a) propagation along the $x$-axis;

$$k = \begin{bmatrix} \frac{2\pi}{\lambda_0} \\ 0 \\ 0 \end{bmatrix} \implies \psi[\mathbf{r},t] = A_0 \cos \left( \frac{2\pi}{\lambda_0} (x - \lambda_0 \nu_0 t) \right) = A_0 \cos \left( \frac{2\pi}{\lambda_0} (x - v_\phi t) \right)$$

(b) propagation along the line $x = y; z = 0$;

$$k = \begin{bmatrix} \frac{2\pi}{\lambda_0} \cos \left( \frac{\pi}{4} \right) \\ \frac{2\pi}{\sqrt{2}\lambda_0} \sin \left( \frac{\pi}{4} \right) \\ 0 \end{bmatrix} \implies \psi[\mathbf{r},t] = A_0 \cos \left[ \frac{2\pi}{\lambda_0} \left( \frac{x + y}{\sqrt{2}} \pm v_\phi t \right) \right]$$

(c) propagation perpendicular to the planes $x + y + z = k$ where $k$ is a constant.

$$\mathbf{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ where } k_x = k_y = k_z \equiv a \implies \mathbf{k} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} \text{ and } |\mathbf{k}| = \frac{2\pi}{\lambda_0} = \sqrt{3}a$$

$$\implies a = \frac{2\pi}{\sqrt{3}\lambda_0}$$

$$\implies \psi[\mathbf{r},t] = A_0 \cos \left( \frac{2\pi}{\sqrt{3}\lambda_0} (x + y + z \pm v_\phi t \cdot \sqrt{3}) \right)$$
6. Two waves of the same amplitude, speed, and frequency travel together in the same region of space. The resultant wave may be written as a sum of two individual waves:

\[ \psi[z,t] = A_0 \sin [k_0z + \omega_0t] + A_0 \sin [k_0z - \omega_0t + \pi] \]

With the help of complex exponentials, show that:

\[ \psi[z,t] = 2A_0 \cos [k_0z] \cdot \sin [\omega_0t] \]

\[
A_0 \sin [k_0z + \omega_0t] + A_0 \sin [k_0z - \omega_0t + \pi] = A_0 \sin [k_0z + \omega_0t] - A_0 \sin [k_0z - \omega_0t]
\]

\[
= A_0 \Im \{\exp [+i (k_0z + \omega_0t)]\} - A_0 \Im \{\exp [+i (k_0z - \omega_0t)]\}
\]

\[
A_0 \Im \{\exp [+ik_0z] (\exp [+i\omega_0t] - \exp [-i\omega_0t])\}
\]

\[
= A_0 \Im \{\exp [+ik_0z] \cdot 2i \sin [+\omega_0t]\}
\]

\[
= 2A_0 \sin [+\omega_0t] \Im \{i \cdot \exp [+ik_0z]\}
\]

\[
= 2A_0 \sin [+\omega_0t] \Im \{i \cdot (\cos [+k_0z] + i \sin [+k_0z])\}
\]

\[
= 2A_0 \sin [+\omega_0t] \Im \{(i \cdot \cos [+k_0z] - \sin [+k_0z])\}
\]

\[
= 2A_0 \cos [k_0z] \cdot \sin [\omega_0t]
\]