

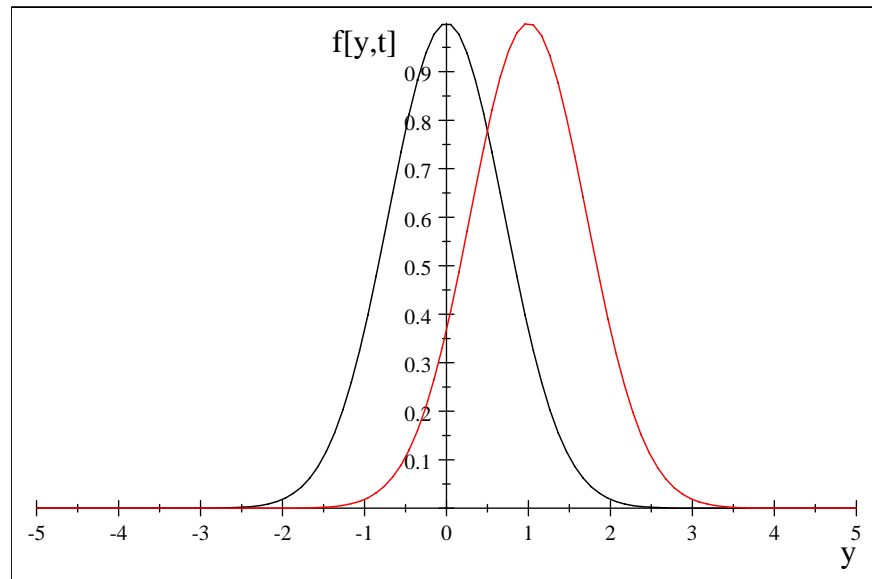
1. Determine which of the following describe traveling waves. Where appropriate, draw the profile and find the speed and direction of motion.

(a)  $\psi [y, t] = \exp [-(a^2y^2 + b^2t^2 - 2abty)]$

*the argument of a traveling wave must have the form  $\Phi [y, t] = k_0z - \omega_0t$  where  $k_0$  and  $\omega_0$  are constants. In this case*

$$\begin{aligned} f [y, t] &= \exp [-(a^2y^2 + b^2t^2 - 2abty)] \\ &= \exp [-(ay - bt)^2] = \exp [-(\Phi [y, t])^2] \end{aligned}$$

*The argument  $u = ay - bt$  does have the proper form for the function  $f [u] = \exp [-u^2]$ , so this is a traveling wave. The argument remains constant for larger  $y$  if  $t$  increases, so the form moves to the right towards  $+\infty$ . Examples are plotted below for different times  $t_0$  and  $t_1 > t_0$  for  $a = b = 1$*



$f [y, t = 0]$  (black) and  $f [y, t = 1]$  (red) for  $a = b = 1$

(b)  $\psi [z, t] = A \sin [az^2 - bt^2]$

we can check this one by applying it to the wave equation  $\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$ :

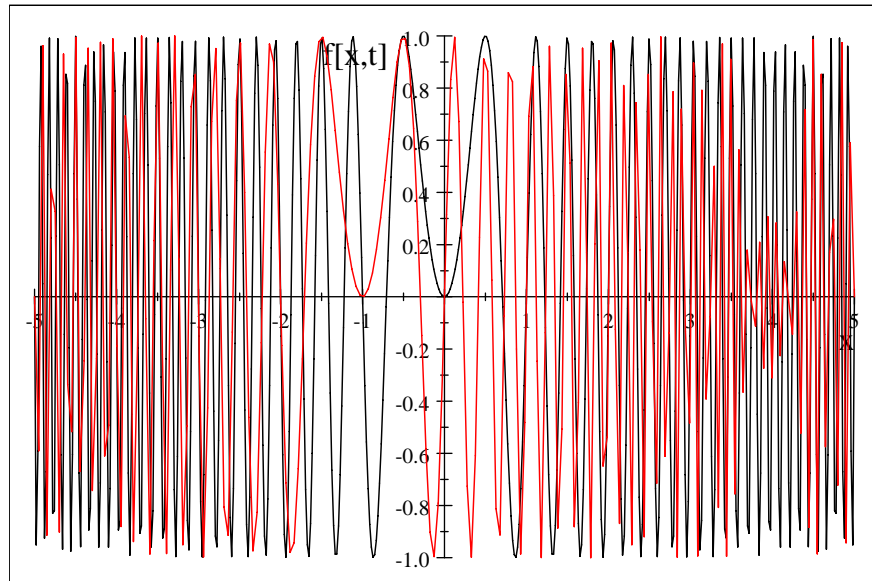
$$\implies v^2 = \frac{\left(\frac{\partial^2 \psi}{\partial t^2}\right)}{\left(\frac{\partial^2 \psi}{\partial z^2}\right)}$$

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= A \frac{\partial}{\partial z} \sin [az^2 - bt^2] = 2Aaz \cos (bt^2 - az^2) \\ \frac{\partial^2 \psi}{\partial z^2} &= \frac{\partial}{\partial z} (2Aaz \cos (bt^2 - az^2)) = 2Aa \cos (bt^2 - az^2) + 4Aa^2 z^2 \sin (bt^2 - az^2) \\ \frac{\partial \psi}{\partial t} &= A \frac{\partial}{\partial t} \sin [az^2 - bt^2] = -2Abt \cos (bt^2 - az^2) \\ \frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial}{\partial t} (-2Abt \cos (bt^2 - az^2)) = 4Ab^2 t^2 \sin (bt^2 - az^2) - 2Ab \cos (bt^2 - az^2) \\ \frac{\left(\frac{\partial^2 \psi}{\partial t^2}\right)}{\left(\frac{\partial^2 \psi}{\partial z^2}\right)} &= \frac{4Ab^2 t^2 \sin (bt^2 - az^2) - 2Ab \cos (bt^2 - az^2)}{4Aa^2 z^2 \sin (bt^2 - az^2) + 2Aa \cos (bt^2 - az^2)} \end{aligned}$$

which does not reduce to a velocity, so NOT a traeveling wave.

(c)  $\psi [x, t] = A \sin \left[ 2\pi \left( \frac{x}{a} + \frac{t}{b} \right)^2 \right]$

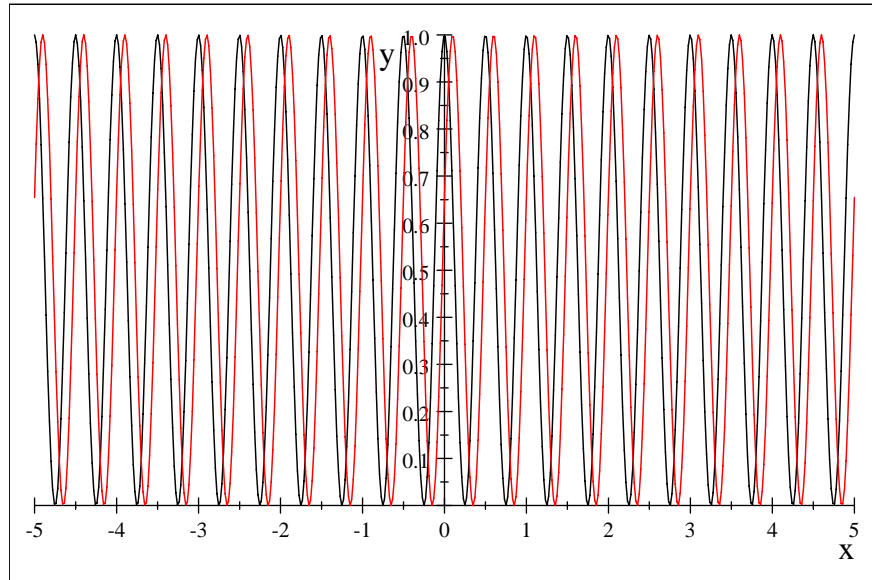
This has the proper form where the velocity is  $v = \lambda_0 \nu_0 = \frac{a}{b}$ . The argument shows that  $x$  must decrease as  $t$  increases, so the function moves to the left towards  $x = -\infty$ . The function is shown below for  $A = 1$  unit,  $a = 1$  unit, and  $b = 1$  sec for  $t_0 = 0$  and  $t_1 = 1$  :



$f [y, t = 0]$  (black) and  $f [y, t = 1]$  (red) for  $A = a = b = 1$  unit

(d)  $\psi [x, t] = A \cos^2 [2\pi (t - x)]$

Again has the correct form so it is a travelling wave. As  $t$  increases, so must  $x$  so this wave moves to the right towards  $x = +\infty$ . Two plots at different times are shown below for  $A = 1$  evaluated at  $t = 0$  (black) and  $t = 0.1$  (red)



$f [y, t = 0]$  (black) and  $f [y, t = 0.1]$  (red) for  $A = 1$  unit

2. The figure shows the profile of a transverse wave on a string traveling in the positive  $z$ -direction at a speed of  $1 \frac{\text{m}}{\text{s}}$ .

(a) Determine its wavelength.

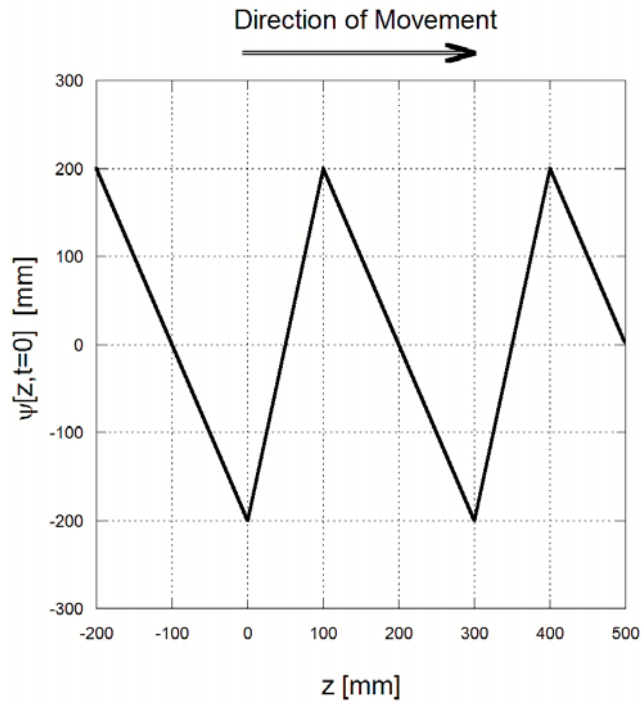
*the distance between adjacent maxima or minima is  $Z_0 = 300 \text{ mm}$*

(b) Notice that as the wave passes any fixed point on the  $z$ -axis, the string at that location oscillates in time. Draw a graph of  $\psi [t]$  showing how a point on the rope at  $z = 0$  oscillates.

*since the velocity is  $1 \frac{\text{m}}{\text{s}}$ , then the function moves past the viewer at one location through  $3\frac{1}{3}$  cycles in one second. The graph of the temporal behavior looks just like that of the spatial behavior except the horizontal axis becomes time in seconds and the unit of  $z = 300 \text{ mm}$  becomes  $t = \frac{3}{10} \text{ sec}$ .*

(c) What is the temporal frequency of the wave?

*As just shown, a wave with period  $\lambda_0 = 300 \text{ mm}$  that moves at  $v = 1 \frac{\text{m}}{\text{s}}$  has one cycle go past in  $\frac{3}{10} \text{ sec}$ , which means that the frequency is  $\frac{10}{3} \text{ Hz}$*



3. A particle executing simple harmonic motion given by

$$y [t] = 4 \sin \left[ 2\pi \frac{t}{6} + \phi_0 \right]$$

is displaced by +1 unit when  $t = 0$ . Find:

(a) the phase angle  $\Phi [t = 0] \equiv \phi_0$

$$\begin{aligned} y [0] &= +1 = 4 \sin \left[ 2\pi \frac{0}{6} + \phi_0 \right] = 4 \sin [\phi_0] \\ \implies \sin [\phi_0] &= +\frac{1}{4} \implies \phi_0 = \sin^{-1} \left[ \frac{1}{4} \right] \\ &\cong 0.252 \text{ radians} \cong 14.4^\circ \cong 14^\circ 24' \\ \implies &\boxed{y [t] = 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right]} \cong 4 \sin \left[ 2\pi \frac{t}{6} + 0.252 \right] \end{aligned}$$

(b) the difference in phase between any two positions of the particle separated in time by 2 sec;

$t_1$  &  $t_2$  are any two times such that  $t_2 - t_1 = 2 \text{ sec}$

$$\begin{aligned} y [t_2] &= 4 \sin \left[ 2\pi \frac{t_2}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] \\ y [t_1] &= 4 \sin \left[ 2\pi \frac{t_1}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] \\ \Delta\Phi &= \Phi [t_2] - \Phi [t_1] = 2\pi \frac{t_2 - t_1}{6} = 2\pi \frac{2 \text{ sec}}{6 \text{ sec}} = \boxed{\frac{2\pi}{3} \text{ radians} = 120^\circ} \end{aligned}$$

(c) the phase angle corresponding to a displacement of +2;

if  $y = +2$  units, then the initial equation is:

$$\begin{aligned} +2 &= 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] = 4 \sin [\Phi [t]] \\ \implies \sin [\Phi [t]] &= \frac{2}{4} \implies \boxed{\Phi [t] = \frac{\pi}{6} = 30^\circ} \end{aligned}$$

(d) the time necessary to reach a displacement of +3 units from the initial position.

if  $y = +3$  units, then the initial equation is:

$$\begin{aligned} +3 &= 4 \sin \left[ 2\pi \frac{t}{6} + \sin^{-1} \left[ \frac{1}{4} \right] \right] = 4 \sin [\Phi [t]] \\ \implies \sin [\Phi [t]] &= \frac{3}{4} \implies \Phi [t] = \sin^{-1} \left[ \frac{3}{4} \right] \cong 48^\circ 40' \\ \implies 2\pi \frac{t}{6} + 14^\circ 24' &\cong 48^\circ 40' \\ \implies 2\pi \frac{t}{6} &\cong 34^\circ 16' \cong 0.598 \implies t \cong \frac{6}{2\pi} \cdot 0.598 \cong \boxed{0.57 \text{ sec}} \end{aligned}$$

4. A wave vibrates according to the equation

$$y [z, t] = \frac{1}{2} \sin \left[ \frac{\pi z}{3} \right] \cdot \cos [40\pi t]$$

where  $y$  and  $z$  are expressed in mm and  $t$  in sec.

- (a) What are the amplitudes and the velocities of the component waves that give rise to this vibration?

*This is a standing wave that may be created by adding two waves with the same amplitude, wavelength, and frequency headed in opposite directions via the relationship:*

$$A_0 \cos [\Phi_1 [z, t]] + A_0 \cos [\Phi_2 [z, t]] = 2A_0 \cos \left[ \frac{\Phi_1 [z, t] + \Phi_2 [z, t]}{2} \right] \cdot \cos \left[ \frac{\Phi_1 [z, t] - \Phi_2 [z, t]}{2} \right]$$

$$\implies A_0 \cos [k_0 z - \omega_0 t] + A_0 \cos [k_0 z + \omega_0 t] = 2A_0 \cos [k_0 z] \cdot \cos [\omega_0 t]$$

so

$$2A_0 = \frac{1}{2} \text{ mm} \implies A_0 = \frac{1}{4} \text{ mm}$$

$$k_0 = \frac{\pi}{3} = \frac{2\pi}{6} \implies \lambda_0 = 6 \text{ mm}$$

$$\omega_0 = 40\pi \implies \nu_0 = 20 \text{ Hz} \implies T_0 = \frac{1}{20} \text{ sec}$$

$$v_\phi = \frac{\omega_0}{k_0} = \lambda_0 \nu_0 = 6 \text{ mm} \cdot 20 \text{ Hz} = \boxed{120 \frac{\text{mm}}{\text{sec}}}$$

- (b) What is the distance between the nodes?

$$\sin \left[ \pi \frac{z}{3} \right] = \sin \left[ 2\pi \frac{z}{6} \right] \implies 2 \text{ nodes in } 6 \text{ mm} \implies \boxed{3 \text{ mm}}$$

- (c) What is the velocity of a particle at the position  $z = 1.5 \text{ mm}$  when  $t = \frac{9}{8} \text{ sec}$ ?

$$y \left[ z = 1.5 \text{ mm}, t = \frac{9}{8} \text{ sec} \right] = \frac{1}{2} \sin \left[ \frac{\pi \cdot 1.5}{3} \right] \cdot \cos \left[ 40\pi \cdot \frac{9}{8} \right]$$

$$= \frac{1}{2} \sin \left[ \frac{\pi}{2} \right] \cdot \cos [35\pi]$$

$$= \frac{1}{2} \cdot 1 \cdot \cos [\pi] = -\frac{1}{2}$$

this is an extremum, so  $\boxed{\text{velocity} = 0 \frac{\text{mm}}{\text{sec}}}$

5. By finding appropriate relations for  $\underline{\mathbf{k}} \bullet \underline{\mathbf{r}}$ , write equations describing a sinusoidal plane wave in three directions in terms of wavelength and velocity for the three cases:

(a) propagation along the  $x$ -axis;

$$\underline{\mathbf{k}} = \begin{bmatrix} \frac{2\pi}{\lambda_0} \\ 0 \\ 0 \end{bmatrix} \implies \psi[\underline{\mathbf{r}}, t] = A_0 \cos \left[ \frac{2\pi}{\lambda_0} (x - \lambda_0 v_\phi t) \right] = A_0 \cos \left[ \frac{2\pi}{\lambda_0} (x - v_\phi t) \right]$$

(b) propagation along the line  $x = y$ ;  $z = 0$ ;

$$\underline{\mathbf{k}} = \begin{bmatrix} \frac{2\pi}{\lambda_0} \cos \left[ \frac{\pi}{4} \right] \\ \frac{2\pi}{\lambda_0} \sin \left[ \frac{\pi}{4} \right] \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2\pi}{\sqrt{2}\lambda_0} \\ \frac{2\pi}{\sqrt{2}\lambda_0} \\ 0 \end{bmatrix} \implies \psi[\underline{\mathbf{r}}, t] = A_0 \cos \left[ \frac{2\pi}{\lambda_0} \left( \frac{x+y}{\sqrt{2}} \pm v_\phi t \right) \right]$$

(c) propagation perpendicular to the planes  $x + y + z = k$  where  $k$  is a constant.

$$\begin{aligned} \underline{\mathbf{k}} &= \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ where } k_x = k_y = k_z \equiv a \implies \underline{\mathbf{k}} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} \text{ and } |\underline{\mathbf{k}}| = \frac{2\pi}{\lambda_0} = \sqrt{3}a \\ \implies a &= \frac{2\pi}{\sqrt{3}\lambda_0} \\ \implies \psi[\underline{\mathbf{r}}, t] &= A_0 \cos \left[ \frac{2\pi}{\sqrt{3}\lambda_0} (x + y + z \pm v_\phi t \cdot \sqrt{3}) \right] \end{aligned}$$

6. Two waves of the same amplitude, speed, and frequency travel together in the same region of space. The resultant wave may be written as a sum of two individual waves:

$$\psi [z, t] = A_0 \sin [k_0 z + \omega_0 t] + A_0 \sin [k_0 z - \omega_0 t + \pi]$$

With the help of complex exponentials, show that:

$$\psi [z, t] = 2A_0 \cos [k_0 z] \cdot \sin [\omega_0 t]$$

$$\begin{aligned} A_0 \sin [k_0 z + \omega_0 t] + A_0 \sin [k_0 z - \omega_0 t + \pi] &= A_0 \sin [k_0 z + \omega_0 t] - A_0 \sin [k_0 z - \omega_0 t] \\ &= A_0 \operatorname{Im} \{ \exp [+i (k_0 z + \omega_0 t)] \} - A_0 \operatorname{Im} \{ \exp [+i (k_0 z - \omega_0 t)] \} \end{aligned}$$

$$\begin{aligned} &A_0 \operatorname{Im} \{ \exp [+ik_0 z] (\exp [+i\omega_0 t] - \exp [-i\omega_0 t]) \} \\ &= A_0 \operatorname{Im} \{ \exp [+ik_0 z] \cdot 2i \sin [+ \omega_0 t] \} \\ &= 2A_0 \sin [+ \omega_0 t] \operatorname{Im} \{ i \cdot \exp [+ik_0 z] \} \\ &= 2A_0 \sin [+ \omega_0 t] \operatorname{Im} \{ i \cdot (\cos [+k_0 z] + i \sin [+k_0 z]) \} \\ &= 2A_0 \sin [+ \omega_0 t] \operatorname{Im} \{ (i \cdot \cos [+k_0 z] - \sin [+k_0 z]) \} \\ &= 2A_0 \cos [k_0 z] \cdot \sin [\omega_0 t] \end{aligned}$$