# 1 Laboratory Reports

Physical Optics 455 - 20063
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## 1.1 Introduction

As a scientist, probably the most fundamental means you will use to communicate with colleagues is through reports and papers that summarize your research. The material you will write about is often complex and non-intuitive. Because of this fact, it is very important that you learn to write these reports clearly and concisely. To do this well often requires significant practice. Laboratory reports are one of the more important opportunities to develop in this area.

In real research, the utility of an experiment is to test some hypothesis or theoretical model. In the labs you will implement this quarter, the theory is well established. However, it will be very useful for you to approach these experiments from the point of view that the theory is not well known; after all, this may well be your first introduction to the theory.

The result of your experiments will fall into one of three categories:

1. the data support the theory within the experimental uncertainties,
2. the data do not support the theory within the experimental uncertainties, or
3. the experimental uncertainties do not allow for a meaningful comparison between the data and the theory.

Conclusion (3) is often, if not usually, a disappointment, because it means that one has to either admit severe deficiencies in the data-taking technique or to revisit the experimental method used and search for a better way to go about making the measurement (of course, this is a very common outcome in real experiments!) However, an honest researcher will report this when necessary. Each week, try to answer the question: into which of these three categories does your experiment fall?

## 1.2 Laboratory Notebook

Students will perform the experiments in teams. Each laboratory team will keep a laboratory notebook that contains descriptions and diagrams of the equipment used in each experiment setup, lists of all data collected, all computations made from those data, etc.

Traditionally, laboratory notebooks are kept in a bound paper book with numbered pages that is made expressly for the purpose, but these days it is completely appropriate in a classroom (rather than industrial) setting to keep an electronic notebook on a Wiki or as a word processor document (e.g., Microsoft Word™). In either case, the electronic notebook should include all data pasted from a spreadsheet (Microsoft Excel™) and diagrams from a sketching/drawing program. For the latter, I use Microsoft Powerpoint™ to make the drawings and then copy the images to Adobe Photoshop™ (via "CTRL-C" to copy the drawing, then make a JPEG image file via "CTRL-N" and paste via "CTRL-V" in Photoshop™). All necessary software for adding images and diagrams to the Wiki lab notebook is available on the computers in the labs. You should keep the lab notebook up to date as you are doing the experiment and NOT wait until after the lab period to fill in the data.

You are probably used to the expectation that you write a laboratory report for each experiment. We are going to take a different tack in this course, where you keep a more complete laboratory notebook and write up only two experiments: one of your choice and one of the instructor's choice. The format of the laboratory reports is provided next.

## 1.3 Submitted Laboratory Reports

Though each team will keep one laboratory notebook, each student will electronically submit TWO individual laboratory reports. The laboratories to be reported will include one each selected by the student and by the instructor.

In your reports, remember the three C's: *concise, clear*, and *complete.*

The format for a laboratory report includes:

- **Experiment Title**
- **Your Name**
- **(Lab Partners' Names)**
- **Date of Experiment:**
- **Introduction or Abstract:**

Brief overview of the experiment providing any relevant background information, the reason for the experiment, what you did, and any conclusions you reached.

- **Experimental Method:**

Include a description of the equipment used with diagrams (not drawn by hand and not copied from the lab descriptions, these drawings are to be made by the student). Include a discussion of how the experiment was performed in a concise summary in your own words, rather than merely repeating the list of instructions provided in the writeup.

- **Data, Analysis, and Discussion:**

In each lab procedure handout, there will be a few questions that you will need to answer. It usually is convenient to answer these questions in the conclusion. All data tables, graphs and calculations are to be integrated into the text rather than included as appendices at the end of the document. You should include tables of all of raw data with units and estimated measurement errors recorded during the experiment as well as all calculations.

The details of your analysis and calculations must be explained. All equations for the calculations are included along with a sample calculation.

All graphs must include the following items:

1. Title;
2. Labeled axes (WITH UNITS SPECIFIED);
3. Data points plotted must be shown with symbols and no connecting lines (the size of the data-point symbol should be chosen wisely);
4. Fits to data and/or theoretical curves should use lines with no symbols;
5. Computed coefficients of any fits of curves to data must be displayed on the plot;
6. Legends should be used if plotting more than one data set on a single graph.

You may use the graphing capabilities in Microsoft Excel™, but recognize that its default format is not appropriate for most graphs. Please, please, PLEASE use white backgrounds in your graphs by turning off the default "gray" background. This leads to the general request that you print dark lines on white backgrounds, period, to minimize toner usage and to allow comments on to be written on the graphs by the grader.

Discuss your results and how they relate to theory. You need to compute the percent difference between the measured and theoretical values. Consider experimental errors and try and determine their most likely source. Try to determine the most significant sources of possible experimental error rather than just listing all possible error sources.

Diagrams of setups are particularly useful in lab reports. These can be very simple – they do not have to be artistic. Include relevant dimensions that are necessary in any equations in the description.

All required material in your lab writeup should be typed on a word processor, both for ease of submission and for archiving. This includes equations, figures, data tables, and answers to any questions. Handwritten pages will not be accepted. Include captions with figures and data tables. Number your equations, data tables, and figures and refer to them by number, e.g., "As demonstrated by Eq. 1, ...", "As shown in Fig. 2, ...", etc. Equations are typically numbered in parentheses located flush with the right margin, while figures and tables are numbered in their captions. The lists of equations, figures, and tables are numbered separately.

The general idea of the lab reports this quarter is to follow the format of real research papers as closely as possible. If you would like some examples of what such papers really look like, go to the library or the CIS Reading Room and look through a couple of imaging science journals, e.g. **Optical Engineering**, the **Journal of the Optical Society of America** (JOSA), or the **Journal of Imaging Science and Technology**.

- **Summary:**

Finally summarize your findings and comment on your success (or lack of) in performing the experiment.

## 1.4 Length of Laboratory Reports

There is no standard length to a lab report. To use a cliché, the lab report should be as long as it must be, but no longer. You need to explain the concepts and the process sufficiently, but do not write pages of description when a few words and/or a figure will suffice.

## 1.5 Grammar and Syntax

Use whole sentences with appropriate grammar and syntax. Please do not use colloquial or slang terms. Also, PROOFREAD your reports before submission – and I don't mean just use the spell check in your word processor. **You are practicing for your profession, so take some pride in the results of your efforts and submit the best report that you can.**

## 1.6 Equations

You will need to include equations and subscripts in your lab writeups, so you need to have the means to do so. Subscript fonts are available in most word processors and equation fonts in many. For example, Microsoft Word$^{TM}$ includes a rudimentary "Equation Editor", and add-on software (such as MathType$^{TM}$ from MathSoft) also is available. You might consider investing in a scientific word processor that includes equation, graphing, and curve-fitting features — many are available, and the time you are likely to save over the course of your college career likely will easily outweigh the cost and learning time.

As already mentioned, number any equations used in your lab writeup. The necessary symbols are available in most symbol fonts and you should be using them in all of your documents and slide presentations. One of my personal peeves is when people substitute the letter "x" for the multiplication symbol "$\times$."

# 2 Measurements and Error in Experiments

## 2.1 The Certainty of Uncertainty

Every measurement exhibits some associated uncertainty. A good experimentalist will try to evaluate the level of uncertainty, which is specified along with the measured value (e.g., $10.4 \pm 0.1\,\text{mm}$), which are then graphed as "error bars" along with the measured numerical value). A very useful reference book on this subject is Data Reduction and Error Analysis for the Physical Sciences (Third Edition), by Philip Bevington and D. Keith Robinson (McGraw-Hill, 2002, ISBN 0072472278). It is not unreasonable to say that this classic book should be on the shelf of every scientist.

In these labs, it is very important for you to estimate the uncertainty. In particular, it is helpful to decide which of the three possible categories mentioned in the introduction that applies to your data. How can you do this? One way is to make what you think is a reasonable decision about the uncertainty. For example, if you measure a length with a ruler, maybe you can only measure the length to some fraction of the smallest rule division. Another way is to measure the uncertainty. That is, take the same measurement multiple times. The average value is then used as the final measurement, and the uncertainty is related to the standard deviation of the individual values. We will discuss these ideas more thoroughly as the class goes on, but you should be prepared to estimate uncertainties for as many measurements as you can. The measurements are usually expressed as the mean value $\mu$ plus or minus ($\pm$) the uncertainty, which generally is standard deviation $\sigma$ of the measurement. Be sure to specify the units used in any measurements! Also watch the number of significant figures; just because your spreadsheet calculates to 7 or 8 places does not mean that this level of precision is merited!

## 2.2 Accuracy vs. Precision

The "accuracy" of an experiment refers to how closely the result came to the "true" value. The "precision" measures how exactly the result was determined, and thus how reproducible the result is.

## 2.3 Significant Figures and Round-Off Error

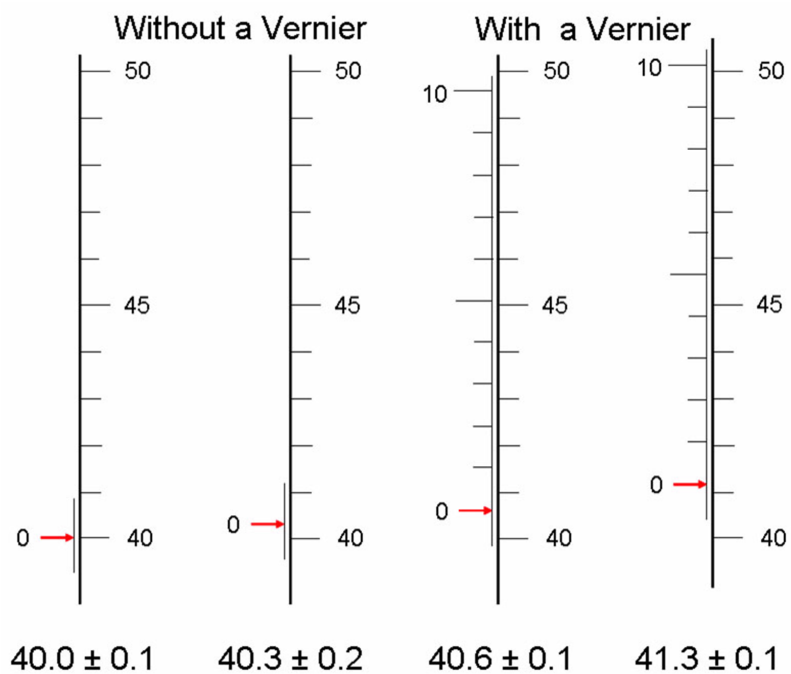
You have probably already learned the definition of the number of significant figures:

1. The MOST significant digit is the leftmost nonzero digit in the number
2. If there is no decimal point, then the LEAST significant digit is the rightmost nonzero digit
3. If there is a decimal point, then the LEAST significant digit is the rightmost digit, even if it is a zero.
4. All digits between the most and least significant digit are "significant digits."

In the result of a measurement, the number of significant digits should be one larger than the precision of any analog measuring device. In other words, you should be able to read an analog scale with more precision than given by the scale; if the scale is labeled in millimeters, you should be able to estimate the measurement to about a tenth of a millimeter. Retain this extra digit and include an estimate of the error, e.g., for a measurement of a length $s$, you might report the measurement as $s = 10.3\,\text{mm} \pm 0.2\,\text{mm}$

## 2.4 Reading Vernier Scales

Vernier scales allow measurements to better precision without having to estimate increments. The mountings of many optical components have such vernier scales to measure lengths or angles. The index of a simple scale (shown in red on the left of each example presented below) is just a single mark that is compared to the adjacent moveable scale (on the right in each example) to obtain the reading. The vernier scale is a set of divisions beyond the index marking on the left laid out with 10 units spanning the same distance as 9 units on the moveable scale on the right, as shown in the example.




*Four readings of a scale. The first two scales do not have a vernier. The red index of the first example on the left is nearly lined up with the value of* 40 *and the uncertainty is probably of the order* $\pm 0.1$ *unit. In the second example, the red index is between* 40 *and* 41*, and the measurement is approximately* $40.3 \pm 0.2$ *units. In the third example, with a vernier, the index is between* 40 *and* 41 *and the sixth line of the vernier scale lines up with one line on the stationary scale; the reading is* $40.6 \pm 0.1$*. In the last example, the index is between* 41 *and* 42 *and the vernier index lines up with the* 3 *on the stationary scale, so the reading is* $41.3 \pm 0.1$*.*

## 3 Propagation of Uncertainty (Error)

Even after you have assessed the uncertainty of your measurements, you still need to estimate the uncertainty of any derived results. For example, if your measurements of "$x$" is 10 units $\pm 1.5$ and "$y$" is 5 units $\pm 0.5$, then what is the undercertainty in the calculation of the sum $z_1 = x + y$? or the ratio $z_2 = \frac{x}{y}$? You should not report precision above and beyond what is warranted. For example, I often see students report 5+ decimal places of precision in numbers derived from observations measured with an uncertainty of more than 10%. This is ridiculous (to put it mildly!). So the next question to consider is how to propagate uncertainties in measurements within calculations made from those measurements. In other words, we need to determine the error in a calculation where pairs of measurements with known precisions are added, subtracted, multiplied, or divided. Suppose that we need to determine a quantity $z$ from other measurements, say $x$ and $y$ via:

$$z = f\left[x, y, \cdots\right]$$

It is generally assumed that the mean value of the quantity $z$ to be determined is the same function of the mean values of the measured quantities:

$$\bar{z} = f\left[\bar{x}, \bar{y}, \cdots\right]$$

The uncertainty in $z$ may be evaluated by considering the variation in $z$ that results from combining individual measurements $x_n$ and $y_n$ via:

$$z_n = f[x_n, y_n, \cdots]$$

The variance in the measurement of $z$ is the average value of the squared magnitude of the difference of the measurement from the mean value:

$$\sigma_z^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N}(z_n - \bar{z})^2\right]$$

which (clearly) has units of the *square* of the calculated quantity $z$. The difference of the $n^{th}$ calculation of $z$ from the mean $\bar{z}$ may be calculated from the differences of the individual measurements from their means:

$$z_n - \bar{z} \cong (x_n - \bar{x})\frac{\partial z}{\partial x} + (y_n - \bar{y})\frac{\partial z}{\partial y} + \cdots$$

Thus the variance in the calculation is:

$$\begin{aligned}
\sigma_z^2 &\cong \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N}\left((x_n - \bar{x})\frac{\partial z}{\partial x} + (y_n - \bar{y})\frac{\partial z}{\partial y} + \cdots\right)^2\right] \\
&= \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N}(x_n - \bar{x})^2\left(\frac{\partial z}{\partial x}\right)^2\right] + \lim_{N\to\infty}\left[+\frac{1}{N}\sum_{n=1}^{N}(y_n - \bar{y})^2\left(\frac{\partial z}{\partial y}\right)^2\right] \\
&\quad + \lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N}2(x_n - \bar{x})(y_n - \bar{y})\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\right] + \cdots
\end{aligned}$$

which may be rewritten in the form:

$$\sigma_z^2 \cong \sigma_x^2\left(\frac{\partial z}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2\sigma_{xy}^2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \cdots$$

The error in the calculation of $z$ is the *standard deviation* $\sigma_z = \sqrt{\sigma_z^2}$ and has the same units as $z$.

Now consider the specific cases that are faced in calculations made from data in experiments.

### 3.1 Error of a Summation or Difference

If the desired measurement $z$ is the weighted sum of two measurements $ax \pm by$, then the partial derivatives are:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= a \\
\frac{\partial z}{\partial y} &= \pm b
\end{aligned}$$

and the resulting variance is:

$$\begin{aligned}
\sigma_z^2 &\cong \sigma_x^2\left(\frac{\partial z}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial z}{\partial y}\right)^2 \pm 2ab\sigma_{xy}^2 \\
&= a^2\sigma_x^2 + (\pm b)^2\sigma_y^2 + 2(+a)(\pm b)\cdot\sigma_{xy}^2 \\
&= a^2\sigma_x^2 + b^2\sigma_y^2 \pm 2ab\cdot\sigma_{xy}^2 \\
\implies \sigma_z &= \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2 \pm 2ab\cdot\sigma_{xy}^2}
\end{aligned}$$

### 3.1.1 Example 1:

In the example in the introduction where $x = 10 \pm 1.5$ and $y = 5 \pm 0.5$, where both are measured in the same units (say, mm), then thesum is:

$$z = x + y = 15.0\,\mathrm{mm}$$

The derivatives are:

$$\frac{\partial z}{\partial x} = a = \frac{\partial z}{\partial y} = b = 1$$

Since the measurements are independent, then $\sigma_{xy} = 0$ and the error in the calculation is:

$$\begin{aligned}\sigma_z &= \sqrt{(\pm 1.5\,\mathrm{mm})^2 + (\pm 0.5\,\mathrm{mm})^2} \\ &= \sqrt{2.5}\,\mathrm{mm} \cong 1.58\,\mathrm{mm}\end{aligned}$$

Therefore the sum should be expressed as:

$$x + y = 15.0\,\mathrm{mm} \pm \sqrt{2.5}\,\mathrm{mm} \cong 15.0\,\mathrm{mm} \pm 1.6\,\mathrm{mm}$$

### 3.1.2 Example 2:

For example, consider the calculation of the perimeter of a quadrilateral shape with sides $x$ and $y$ from measurements of two sides; the perimeter is:

$$z = 2x + 2y$$

The derivatives are:

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2 \\ \frac{\partial z}{\partial y} &= 2\end{aligned}$$

and the variance in the calculation is:

$$\begin{aligned}\sigma_z^2 &= \sigma_x^2 \left(\frac{\partial z}{\partial x}\right)^2 + \sigma_y^2 \left(\frac{\partial z}{\partial y}\right)^2 + 2\sigma_{xy}^2 \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) \\ &= 2^2\sigma_x^2 + 2^2\sigma_y^2 + 2 \cdot 2 \cdot 2\sigma_{xy}^2 = 4\left(\sigma_x^2 + \sigma_y^2\right)\end{aligned}$$

If the measurement of $x$ is $100\,\mathrm{mm} \pm 1\,\mathrm{mm}$ and that of $y = 1000\,\mathrm{mm} \pm 10\,\mathrm{mm}$, then the perimeter is:

$$z = 2 \cdot 100\,\mathrm{mm} + 2 \cdot 1000\,\mathrm{mm} = 2200\,\mathrm{mm}$$

The standard deviation in the calculation is:

$$\sigma_z = \sqrt{4 \cdot (\pm 1\,\mathrm{mm})^2 + 4 \cdot (\pm 10\,\mathrm{mm})^2} \cong 20.1\,\mathrm{mm}$$

so we speak of the calculation as:

$$z = 2200\,\mathrm{mm} \pm 20\,\mathrm{mm}$$

What if the error in the measurement of $y$ is the same as that in the measurement of $x$? Then the standard deviation is:

$$\sigma_z = \sqrt{4 \cdot (\pm 1\,\mathrm{mm})^2 + 4 \cdot (\pm 1\,\mathrm{mm})^2} = 2\sqrt{2}\,\mathrm{mm} \cong 2.83\,\mathrm{mm}$$

$$z = 2200\,\mathrm{mm} \pm 3\,\mathrm{mm}$$

If the perimeter is calculated from measurments of the four sides individually, each with standard deviation of 1 mm:

$$z = x_1 + x_2 + y_1 + y_2$$

The variance in the measurement is:

$$\begin{aligned}\sigma_z^2 &= 1^2 \cdot \sigma_{x_1}^2 + 1^2 \cdot \sigma_{x_2}^2 + 1^2 \cdot \sigma_{y_1}^2 + 1^2 \cdot \sigma_{y_2}^2 \\ &= (1\,\text{mm})^2 + (1\,\text{mm})^2 + (1\,\text{mm})^2 + (1\,\text{mm})^2 = 4\,\text{mm}^2 \\ \sigma_z &= \sqrt{4\,\text{mm}^2} \cong 2\,\text{mm}\end{aligned}$$

So the calculation is:

$$z = 2200\,\text{mm} \pm 2\,\text{mm}$$

and the error is less than that in the calculation of the perimeter from measurements of two sides.

## 3.2 Error of a Product or Ratio

If the computation is the scaled product of the two measurements $x$ and $y$, e.g.,

$$z = \pm a \cdot x \cdot y$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \pm a \cdot y \\ \frac{\partial z}{\partial y} &= \pm a \cdot x\end{aligned}$$

Variance of Product:

$$\begin{aligned}\sigma_z^2 &= \sigma_x^2 \left(\pm a \cdot y\right)^2 + \sigma_y^2 \left(\pm a \cdot x\right)^2 + 2\left(\pm a \cdot y\right)\left(\pm a \cdot x\right)\sigma_{xy}^2 \\ &= a^2 y^2 \sigma_x^2 + a^2 x^2 \sigma_y^2 + 2a^2 xy\sigma_{xy}^2 \\ &= a^2\left(y^2\sigma_x^2 + x^2\sigma_y^2 + 2xy\sigma_{xy}^2\right)\end{aligned}$$

whihc may be rewritten as:

$$\begin{aligned}\sigma_z^2 &= \left(a^2x^2y^2\right)\frac{\sigma_x^2}{x^2} + \left(a^2x^2y^2\right)\frac{\sigma_y^2}{y^2} + 2\left(a^2x^2y^2\right)\frac{\sigma_{xy}^2}{xy} \\ \implies \frac{\sigma_z^2}{z^2} &= \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} + 2\frac{\sigma_{xy}^2}{xy}\end{aligned}$$

If the computation is the ratio of the two measurements:

$$\begin{aligned}z &= \pm\frac{ax}{y} \\ \implies \frac{\partial z}{\partial x} &= \pm\frac{a}{y} \\ \implies \frac{\partial z}{\partial y} &= \mp\frac{ax}{y^2}\end{aligned}$$

$$
\begin{aligned}
\sigma_z^2 &= \sigma_x^2\left(\pm\frac{a}{y}\right)^2 + \sigma_y^2\left(\pm\frac{ax}{y^2}\right)^2 + 2\sigma_{xy}^2\left(\pm\frac{a}{y}\right)\left(\mp\frac{ax}{y^2}\right) \\
&= \sigma_x^2\frac{a^2}{y^2} + \sigma_y^2\frac{a^2x^2}{y^4} - 2\sigma_{xy}^2\frac{a^2x}{y^3} \\
&= \frac{\sigma_x^2}{x^2}\left(\frac{a^2x^2}{y^2}\right) + \left(\frac{\sigma_y^2}{y^2}\frac{a^2x^2y^2}{y^4}\right) - 2\frac{\sigma_{xy}^2}{xy}\frac{a^2x^2}{y^2} \\
&= \frac{\sigma_x^2}{x^2}\left(z^2\right) + \frac{\sigma_y^2}{y^2}\left(z^2\right) - 2\frac{\sigma_{xy}^2}{xy}\left(z^2\right) \\
&\Longrightarrow \frac{\sigma_z^2}{z^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} - 2\frac{\sigma_{xy}^2}{xy}
\end{aligned}
$$

Note the similarities between the expressions for the product and for the ratio:

$$
\begin{aligned}
\textit{Product} &: \quad \frac{\sigma_z^2}{z^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} + 2\frac{\sigma_{xy}^2}{xy} \\
\textit{Ratio} &: \quad \frac{\sigma_z^2}{z^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} - 2\frac{\sigma_{xy}^2}{xy}
\end{aligned}
$$

### 3.2.1 Example 3:

Now consider the area of the quadrilateral used in the example 2, where $x = 100\,\text{mm} \pm 1\,\text{mm}$ and $y = 1000\,\text{mm} \pm 10\,\text{mm}$. The calculated area is:

$$z = x \cdot y = 100,000\,\text{mm}^2 = 10^5\,\text{mm}^2$$

The uncertainty is obtained from the derivatives after setting $a = 1$:

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= y \\
\frac{\partial z}{\partial y} &= x
\end{aligned}
$$

$$
\begin{aligned}
\sigma_z &= \sqrt{y^2\sigma_x^2 + x^2\sigma_y^2 + 2xy\sigma_{xy}^2} \\
&= \sqrt{(1000\,\text{mm})^2 \cdot (1\,\text{mm})^2 + (100\,\text{mm})^2 \cdot (10\,\text{mm})^2} \\
&\cong 1414\,\text{mm}^2
\end{aligned}
$$

so the calculation is:

$$z = 100,000\,\text{mm}^2 \pm 1414\,\text{mm}^2$$

If the error in the measurement of $y$ is the same as that of the measurement of $x$ (i.e., $\pm 1\,\text{mm}$), the error in the area is somewhat smaller:

$$z = 100,000\,\text{mm}^2 \pm 1005\,\text{mm}^2$$

## 3.3 Value of a Measurement Raised to a Power

If the calculation is proportional to a measured value raised to a power, such as:

$$z = ax^{\pm b}$$

The derivative of the calculation with respect to the measurement is:

$$\frac{\partial z}{\partial x} = \pm ab \cdot \left(x^{\pm b - 1}\right) = \pm\frac{bz}{x}$$

and the standard deviation is related by:

$$\frac{\sigma_z}{z} = b\frac{\sigma_x}{x}$$

### 3.3.1 Example 4:

Consider the error in the area of a circle based on an inaccurate measurement of its diamter, say $x = 100\,\mathrm{mm}$, $\sigma_x = 5\,\mathrm{mm}$. The area is

$$z = \pi\left(\frac{x}{2}\right)^2 = \frac{\pi}{4}\cdot(100\,\mathrm{mm})^2 = 2500\pi\,\mathrm{mm}^2 \cong 7854.0\,\mathrm{mm}^2$$

which implies that $a = \pi$ and $b = 2$. The standard deviation is:

$$\begin{aligned}\sigma_z = zb\cdot\frac{\sigma_x}{b} &= 2500\pi\,\mathrm{mm}^2\cdot 2\cdot\frac{5\,\mathrm{mm}}{100\,\mathrm{mm}}\\ &= 250\pi\,\mathrm{mm}^2 = 785.40\,\mathrm{mm}^2\end{aligned}$$

so the calculation of the area is:

$$z = 7854.0\,\mathrm{mm}^2 \pm 785.4\,\mathrm{mm}^2$$

## 3.4 Sample Problems:

1. Determine the uncertainty in the calculation of $z$ from the measurement $x$ via the exponential:

$$z = ae^{\pm bx}$$

where $a$ and $b$ are constants.

**Solution:**

$$\frac{\partial z}{\partial x} = abe^{\pm bx} = bz$$

$$\begin{aligned}\sigma_z^2 &\cong \sigma_x^2\left(\frac{\partial z}{\partial x}\right)^2\\ &= \sigma_x^2\cdot(bz)^2\\ \sigma_z &= \sigma_x\cdot|bz|\\ \frac{\sigma_z}{z} &= |b|\,\sigma_x\end{aligned}$$

2. Determine the uncertainty in the calculation of $z$ from the measurement $x$ via a logarithm:

$$z = a\ln(bx)$$

where $a$ and $b$ are constants.

**Solution:**

$$\begin{aligned}z &= a\ln(bx) = a\ln(b) + a\ln(x)\\ \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(a\ln(b) + a\ln(x))\\ &= \frac{\partial}{\partial x}(a\ln(b)) + \frac{\partial}{\partial x}(a\ln(x))\\ &= 0 + a\frac{\partial}{\partial x}(\ln(x)) = \frac{a}{x}\end{aligned}$$

$$\begin{aligned}\sigma_z^2 &\cong \sigma_x^2\left(\frac{\partial z}{\partial x}\right)^2\\ &= \sigma_x^2\left(\frac{a}{x}\right)^2\\ &\Longrightarrow \sigma_z \cong \sigma_x\left|\frac{a}{x}\right| \propto x^{-1}\end{aligned}$$

3. The activity of a radioactive source as a function of time is:

$$A = A_0 \exp\left[-\frac{t}{\tau}\right]$$

where $A_0$ is the initial activity (measured in $\frac{\text{decays}}{\text{sec}}$) and $\tau$ is the *natural lifetime* (i.e., the time after which the activity has reduced by a factor of $e^{-1} \cong 0.368$). Determine the uncertainty in the initial activity of a radioactive source measured after $t = 20$ days$\pm 1$ hour if the initial activity is $A_0 = 1000\frac{\text{decays}}{\text{sec}}$ and $\tau = 5$ days.

**Solution:**

This is an example of the exponential in problem 1:

$$\begin{aligned} z &= ae^{\pm bx} \\ \text{where } a &= A \text{ and } b = \tau^{-1} \end{aligned}$$

$$z \longrightarrow A = 1000\frac{\text{decays}}{\text{sec}} \cdot \exp\left[-\frac{20}{5}\right] \cong 18.32\frac{\text{decays}}{\text{sec}}$$

$$\begin{aligned} \sigma_A &= \sigma_t \cdot |bz| \\ &= 1 \text{ hour} \cdot \left|\frac{1}{5 \text{ days}} \cdot 18.32\frac{\text{decays}}{\text{sec}}\right| \\ &= \frac{1}{120} \cdot 18.32\frac{\text{decays}}{\text{sec}} \cong 0.15\frac{\text{decays}}{\text{sec}} \end{aligned}$$

so the measured activity is:

$$A = 18.32 \pm 0.15\frac{\text{decays}}{\text{sec}}$$