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Additional Comments on Radiometric Topics[1]

1 Plane or Linear Angle

The plane angle, θ is defined as the length of arc of a circle divided by the radius of the circle,

$$\theta = \frac{s}{r} \left[\frac{m}{m} = rad\right] \tag{1}$$

We can see this in terms of the *unit circle* (see Fig. 1). The plane angle is dimensionless. However, to aid in communication, it has been given the Standard International (SI) unit of measure, the *radian*. The radian is just a measure of the angle subtended by a one-dimensional line about the origin in a two-dimensional (plane) space. Subsequently, there are 2π radians in a full circle, given that the length of arc, s encompasses the full unit circle. We know this from basic geometry ($C = 2\pi r = 2\pi$, for a unit circle).

However, an angle is a subtense, a linear dimension divided by another linear dimension. As Fig. 1 shows, a straight line and even a curved line can subtend the same angle as an arc on the circle. So, the plane angle should be defined as: a plane angle is the *projection* of a line on a unit circle, and the line *need not be straight*. The angle is really determined by the end points of the line.

2 Solid Angle

A solid angle is the two-dimensional equivalent of a plane angle. It is the projection of an *area* (or a closed curve in space) onto a unit sphere. The solid angle is determined by the perimeter of the area, A projected on the surface (see Fig. 2).

[1]Submit suggestions and typos to emmett@cis.rit.edu

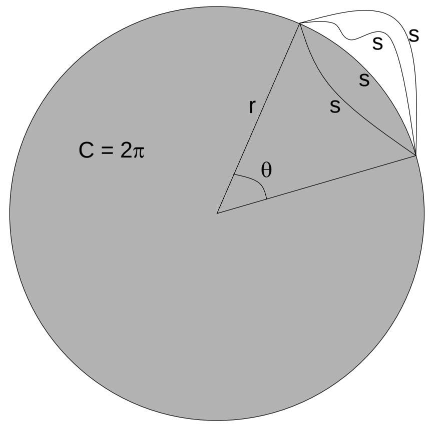

Figure 1: Illustration of plane angle, $\theta = s/r$, for any s.

An equivalent definition of a solid angle is the projection of an area, A onto a sphere divided by the square of the radius of the sphere. In differential form this is expressed as

$$\partial\Omega = \frac{\partial A}{r^2} \; \left[\frac{m^2}{m^2} = sr\right] \tag{2}$$

Important Note: Other than what Fig 2. might suggest, the shape of the area does not matter at all. Any shape on the surface of the sphere that holds the same area will define a solid angle of the same size.

The units of measure are *steradian* which is a measure of the angular area subtended by a two-dimensional surface about the origin in a three-dimensional space. Steradians are equivalently referred to as "square radians." Mathematically, the solid angle, like the radian, is really unitless ($m^2/m^2 = 1$). However, for practical reasons, the steradian, from Greek *stereos* meaning *solid*, is assigned. The unit originated in the 1870s by analogy with the radian.

A sphere subtends 4π square-radians (or steradians) about the origin. Numerically, the number of steradians in a sphere is equal to the surface area, SA of a sphere of unit radius. That is, $SA = 4\pi r^2$, but with $r = 1, SA = 4\pi$.

As a final example, a point source of electromagnetic (EM) power that radiates equally well in all directions (isotropic), and whose output intensity I, is 1 $W sr^{-1}$, has a total output power of 4π (approximately 12.56) watts. This is because there are 4π steradians in three-dimensional space with respect to a point of reference.

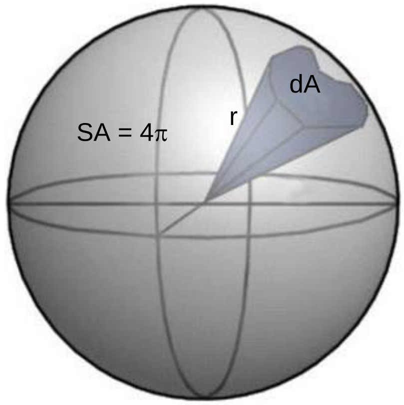

Figure 2: Illustration of solid angle on a unit sphere, $\partial\Omega = \partial A/r^2$, for any A.

3 Units in the Inverse Square Law Equation

There have been many questions as to the units in the relation

$$E = \frac{I}{r^2} \left[\frac{W}{m^2}\right] \tag{3}$$

The units for intensity, I are $[W sr^{-1}]$ while the units for radius squared are $[m^2]$. So what happened to the steradian unit in the expression for irradiance, E? The answer is in the derivation for the relation and in the definition for steradian.

Recall the definitions for irradiance and intensity

$$E = \frac{\partial\Phi_E}{\partial A} \tag{4}$$

$$I = \frac{\partial\Phi_I}{\partial\Omega} \tag{5}$$

If we consider an element of solid angle $\partial\Omega$ encompassing and area element ∂A on a surface of interest while additionally considering a lossless isotropic media, we can let $\partial\Phi_E = \partial\Phi_I = \partial\Phi$. We can write

$$E\ \partial A\ [Wm^{-2}m^2] = \partial\Phi\ [W] = I\ \partial\Omega\ [W sr^{-1} sr] \tag{6}$$

It is at this point we see the units for meters and steradians cancelling out. We then make a final substitution for $\partial\Omega = \partial A/r^2\ [m^2/m^2]$ which yields Eqn. 3. Remember, mathematically, the steradian is unitless.

A units analysis produces,

$$\frac{\frac{W}{sr}}{m^2} = \frac{W}{sr\ m^2} = \frac{W}{\frac{m^2}{m^2}m^2} = \frac{W}{m^2} \tag{7}$$

which are the units for irradiance.

4 Definition of Radiance (and its Variants)

Recall the definitions for irradiance and intensity

$$E = \frac{\partial \Phi}{\partial A} \tag{8}$$

$$I = \frac{\partial \Phi}{\partial \Omega} \tag{9}$$

We would like to generate a new definition, L that captures the essence of both irradiance *and* intensity. To do this, for example, we incorporate the change in angle, $\partial\Omega$ into Eqn. 8,

$$\frac{\partial}{\partial \Omega}(E) = \frac{\partial}{\partial \Omega}\left(\frac{\partial \Phi}{\partial A}\right) = \frac{\partial^2 \Phi}{\partial \Omega\, \partial A} = L_E \tag{10}$$

Additionally we can do the same for intensity by incorporating the change in area, ∂A into Eqn. 9,

$$\frac{\partial}{\partial A}(I) = \frac{\partial}{\partial A}\left(\frac{\partial \Phi}{\partial \Omega}\right) = \frac{\partial^2 \Phi}{\partial A\, \partial \Omega} = L_I \tag{11}$$

If Eqns. 10 and 11 are continuous, then the equality of mixed partial derivatives tells us that $L_E = L_I = L$ which is the definition of radiance. Finally, after factoring in projected area effects we have,

$$\boxed{L = \frac{\partial^2 \Phi}{\partial \Omega\, \partial A\ \cos\theta}} \tag{12}$$

We can re-write the definition of radiance in terms of irradiance. We illustrate this by first taking the partial derivative of irradiance with respect A,

$$\frac{\partial}{\partial A}(E) = \frac{\partial}{\partial A}\left(\frac{\partial \Phi}{\partial A}\right) \tag{13}$$

$$\frac{\partial E}{\partial A} = \frac{\partial^2 \Phi}{\partial A^2} \tag{14}$$

$$\partial^2 \Phi = \frac{\partial A^2\, \partial E}{\partial A} = \partial A\, \partial E \tag{15}$$

Substituting the result from Eqn. 15 into Eqn. 12 we have,

$$L = \frac{\partial A \, \partial E}{\partial \Omega \, \partial A \cos \theta} \tag{16}$$

$$\boxed{L = \frac{\partial E}{\partial \Omega \; \cos \theta}} \tag{17}$$

Similarly, we can generate an expression for radiance in terms of intensity.

$$\frac{\partial}{\partial \Omega}(I) = \frac{\partial}{\partial \Omega}\left(\frac{\partial \Phi}{\partial \Omega}\right) \tag{18}$$

$$\frac{\partial I}{\partial \Omega} = \frac{\partial^2 \Phi}{\partial \Omega^2} \tag{19}$$

$$\partial^2 \Phi = \frac{\partial \Omega^2 \, \partial I}{\partial \Omega} = \partial \Omega \, \partial I \tag{20}$$

Substituting the result from Eqn. 20 into Eqn. 12 we have,

$$L = \frac{\partial \Omega \, \partial I}{\partial \Omega \, \partial A \cos \theta} \tag{21}$$

$$\boxed{L = \frac{\partial I}{\partial A \; \cos \theta}} \tag{22}$$

5 When to use Cosine and Sine

In radiometry problems we often encounter situations when projected area effects need to be taken into account. A question that often arises is when to compute the cosine of the given angle versus the sine. This depends on how the angle is defined in the given problem. For example, in Figure 3, we have an irradiance, E_o falling on a surface that is rotated by angle θ (blue-lower line). Based on this angle, we can form a triangle made up of vectors, E_o, the perpendicular to the illuminated surface, E_{cos}, and a translated vector that sits across from angle θ called E_{sin}. If we are looking for the irradiance onto the rotated surface, then we need to compute the irradiance in the E_{cos} direction. Relative to the angle θ, we have hypotonus E_o and adjacent E_{cos}, which is tells use to use the trigonometric function cosine.

If the surface was rotated in the "forward" direction, as shown in Figure 3 (red-upper line), and the angle θ was define as illustrated, then we would be looking for the irradiance in the E_{sin} direction. Here, we would use the sine function for we have *opposite* vector E_{sin} and *hypotonus* vector E_o.

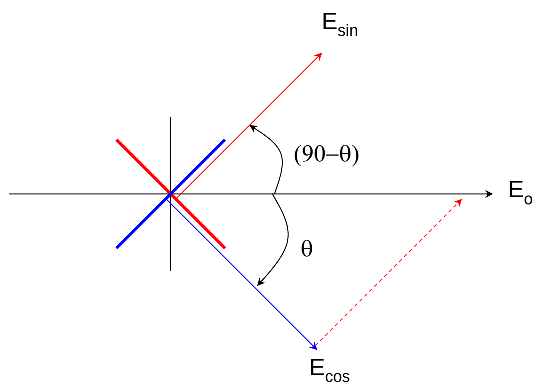

Figure 3: Geometry for calculating the irradiance onto a surface rotated by angles θ and (90 - θ).