1051-320-20062 Solution Set #2 (Corrected 01/13/2007)

1. Find the lengths of the 3-D vectors:

   SOLUTION:

$$\textit{Length} \ \ |\underline{\mathbf{x}}| \equiv \sqrt{\underline{\mathbf{x}} \bullet \underline{\mathbf{x}}}$$

   (a)

$$\begin{aligned}
\underline{\mathbf{x}}_a &= \begin{pmatrix} +1 \\ +1 \\ -1 \end{pmatrix}, \ \underline{\mathbf{x}}_a \bullet \underline{\mathbf{x}}_a = (+1)^2 + (+1)^2 + (-1)^2 = 1 + 1 + 1 = 3 \\
& \boxed{|\underline{\mathbf{x}}_a| = \sqrt{3}} \\
\underline{\mathbf{x}}_b &= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \ \underline{\mathbf{x}}_b \bullet \underline{\mathbf{x}}_b = (-1)^2 + (-1)^2 + (-1)^2 = 1 + 1 + 1 = 3 \\
& \boxed{|\underline{\mathbf{x}}_b| = \sqrt{3}} \\
\underline{\mathbf{x}}_c &= \begin{pmatrix} +1 \\ -1 \\ +1 \end{pmatrix}, \ \underline{\mathbf{x}}_c \bullet \underline{\mathbf{x}}_c = (+1)^2 + (-1)^2 + (+1)^2 = 1 + 1 + 1 = 3 \\
& \boxed{|\underline{\mathbf{x}}_c| = \sqrt{3}}
\end{aligned}$$

   All three vectors have the same length

2. For each pair of vectors $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$, find the scalar product $\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2$:

SOLUTION:

$$\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 \equiv \sum_{n=0}^{2} (\underline{\mathbf{v}}_1)_n \, (\underline{\mathbf{v}}_2)_n$$

(a) $\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 = (+1)\,(+1) + (+1)\,(+1) + (+1)\,(+1) = \boxed{3} \quad \textit{(duh)}$$

(b) $\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{aligned} \underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 &= (+1)\,(0) + (0)\,(+1) + (0)\,(+1) = \boxed{0} \\ & \textit{so } \underline{\mathbf{v}}_1 \perp \underline{\mathbf{v}}_2 \end{aligned}$$

(c) $\underline{\mathbf{v}}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}; \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

$$\begin{aligned} \underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 &= \left(+\frac{1}{\sqrt{3}}\right)\left(+\frac{1}{\sqrt{3}}\right) + (0)\left(-\frac{1}{\sqrt{3}}\right) + \left(+\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) = \boxed{0} \\ & \textit{so again } \underline{\mathbf{v}}_1 \perp \underline{\mathbf{v}}_2 \end{aligned}$$

3. Find the *projection* of the first vector in the direction of the second vector AND the projection of the second vector onto the direction of the first.

$$\begin{aligned} \textit{Projection of } \underline{\mathbf{v}}_2 \textit{ onto } \underline{\mathbf{v}}_1 \textit{ is } \hat{\underline{\mathbf{v}}}_1 \bullet \underline{\mathbf{v}}_2 &= \frac{\underline{\mathbf{v}}_1}{|\underline{\mathbf{v}}_1|} \bullet \underline{\mathbf{v}}_2 \\ \textit{Projection of } \underline{\mathbf{v}}_1 \textit{ onto } \underline{\mathbf{v}}_2 \textit{ is } \hat{\underline{\mathbf{v}}}_2 \bullet \underline{\mathbf{v}}_1 &= \frac{\underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_2|} \bullet \underline{\mathbf{v}}_1 \end{aligned}$$

(a) $\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}; \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$$\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 = (1 \cdot 0 + 2 \cdot (-1) + (-1) \cdot (+1)) = -3$$

$$\begin{aligned} |\underline{\mathbf{v}}_1| &= \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6} \\ |\underline{\mathbf{v}}_2| &= \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2} \end{aligned}$$

$$\textit{Projection of } \underline{\mathbf{v}}_2 \textit{ onto } \underline{\mathbf{v}}_1 = \frac{\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_1|} \boxed{= \tfrac{-3}{\sqrt{6}} = -\tfrac{\sqrt{6}}{2} = -\sqrt{\tfrac{3}{2}} \cong -1.225}$$

(so the projection is in the "opposite" direction of $\underline{\mathbf{v}}_1$*)*

$$\textit{Projection of } \underline{\mathbf{v}}_1 \textit{ onto } \underline{\mathbf{v}}_2 = \frac{\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_2|} = \frac{1}{\sqrt{2}}(-3) = \boxed{-\tfrac{3}{\sqrt{2}} = -\tfrac{3\sqrt{2}}{2} \cong -2.121}$$

(b) $\underline{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}; \quad \underline{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$

$$\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2 = (1 \cdot 0 + 2 \cdot (-2) + (-1) \cdot (+2)) = -6$$

$$\begin{aligned} |\underline{\mathbf{v}}_1| &= \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6} \\ |\underline{\mathbf{v}}_2| &= \sqrt{0^2 + (-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \end{aligned}$$

$$\textit{Projection of } \underline{\mathbf{v}}_2 \textit{ onto } \underline{\mathbf{v}}_1 = \frac{\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_1|} = \frac{1}{\sqrt{6}}(-6) = \boxed{-\sqrt{6} \cong -2.450}$$

(so the projection is twice as long, because $\underline{\mathbf{v}}_2$ *is twice as long as in part a)*

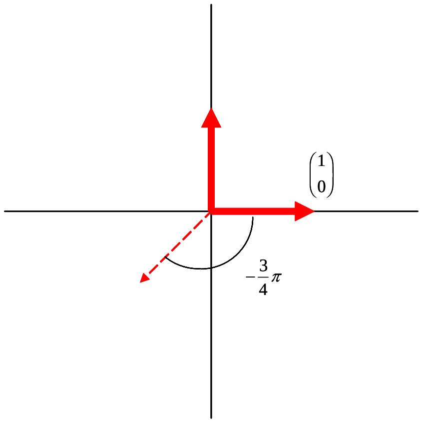
$$\textit{Projection of } \underline{\mathbf{v}}_1 \textit{ onto } \underline{\mathbf{v}}_2 = \frac{\underline{\mathbf{v}}_1 \bullet \underline{\mathbf{v}}_2}{|\underline{\mathbf{v}}_2|} = \frac{1}{2\sqrt{2}}(-6) = \boxed{-\tfrac{3}{\sqrt{2}} = -\tfrac{3\sqrt{2}}{2} \cong -2.121}$$

(so the projection is the same as in part a, because $\underline{\mathbf{v}}_1$ *is the same length as it was there*

4. Find the vectors obtained by rotating the two 2-D basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by two different angles:

   (a) $\theta = -\frac{3\pi}{4}$ radians

$$\cos\left[-\frac{3\pi}{4}\right] = \sin\left[-\frac{3\pi}{4}\right] = -\frac{1}{\sqrt{2}}$$



From the skecth, the first vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is rotated to* $\boxed{\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}}$, *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *to* $\boxed{\begin{pmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}}$. *This is a sufficient answer, BUT we can derive a matrix for performing the rotation:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Longrightarrow a = -\frac{1}{\sqrt{2}}, \; c = -\frac{1}{\sqrt{2}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Longrightarrow b = +\frac{1}{\sqrt{2}}, \; d = -\frac{1}{\sqrt{2}}$$

So the rotation matrix is:

$$\underline{\mathcal{R}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Note that

$$\det \underline{\mathcal{R}} = \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\left(+\frac{1}{\sqrt{2}}\right) = 1$$

The general form of a matrix that rotates a vector by θ is:

$$\boxed{\underline{\mathcal{R}}_\theta = \begin{bmatrix} \cos[\theta] & -\sin[\theta] \\ +\sin[\theta] & \cos[\theta] \end{bmatrix}}$$

and its determinant is:

$$\det \underline{\mathcal{R}}_\theta = \cos^2[\theta] - \sin[\theta] \cdot (-\sin[\theta]) = \cos^2\theta + \sin^2[\theta] = +1$$

The determinant of a rotation matrix is 1

(b) $\theta = +\frac{\pi}{4}$ radians

Use the rotation matrix from part a:

$$\underline{\mathcal{R}}_{\left(+\frac{\pi}{4}\right)} = \begin{bmatrix} \cos\left[+\frac{\pi}{4}\right] & -\sin\left[+\frac{\pi}{4}\right] \\ +\sin\left[+\frac{\pi}{4}\right] & \cos\left[+\frac{\pi}{4}\right] \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and its determinant is:

$$\det \underline{\mathcal{R}}_{\left(+\frac{\pi}{4}\right)} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = 1$$

The rotated vectors are:

$$\underline{\mathcal{R}}_{\left(+\frac{\pi}{4}\right)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}$$

and:

$$\underline{\mathcal{R}}_{\left(+\frac{\pi}{4}\right)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}$$

5. Find k so that the two vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are orthogonal:

$$\underline{\mathbf{a}} = \begin{pmatrix} -2 \\ +5 \\ -4 \end{pmatrix}, \underline{\mathbf{b}} = \begin{pmatrix} 1 \\ k \\ -3 \end{pmatrix}$$

$$\begin{aligned} \underline{\mathbf{a}} \perp \underline{\mathbf{b}} &\implies \underline{\mathbf{a}} \bullet \underline{\mathbf{b}} = 0 \\ \underline{\mathbf{a}} \bullet \underline{\mathbf{b}} &= (-2)(+1) + (+5)(+k) + (-4)(-3) \\ &= 5k + 10 \\ 5k + 10 &= 0 \implies 5k = -10 \implies \boxed{k = -2} \end{aligned}$$

CHECK:

$$\underline{\mathbf{a}} \bullet \underline{\mathbf{b}} = \begin{pmatrix} -2 \\ +5 \\ -4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0$$