Series Expansions You Should Know:

1. *Geometric Series*

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \ \ \textit{if} \ \ |t| < 1$$
$$= 1 + t + t^2 + t^3 + \cdots$$

Examples:

$$\frac{1}{0.9} = \frac{1}{1-0.1} = 1 + 0.1 + 0.01 + 0.001 + \cdots = 1.11111\cdots$$
$$\frac{1}{0.25} = \frac{1}{1-0.75} = 1 + (0.75) + (0.75)^2 + (0.75)^3 + (0.75)^4 + (0.75)^5 + \cdots$$

$$(1+x)^{\frac{1}{2}} = \sqrt{(1+x)} = 1 + \frac{1}{2}\cdot x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\cdot x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\cdot x^2 + \cdots$$
$$= 1 + \frac{1}{2}\cdot x - \frac{1}{8}\cdot x^2 + \frac{1}{16}\cdot x^3 + \cdots$$

2. *Finite Geometric Series:*

$$\sum_{n=0}^{N} t^n = 1 + t + t^2 + t^3 + \cdots + t^N$$
$$= \sum_{n=0}^{\infty} t^n - \sum_{n=N}^{\infty} t^n$$
$$= \left(\frac{1}{1-t}\right) - t^{N-1}\cdot\frac{1}{1-t}$$
$$= \frac{1-t^{N-1}}{1-t}$$

$$\sum_{n=0}^{N} t^n = \frac{1-t^{N+1}}{1-t}$$

Examples:

$$\sum_{n=0}^{4} (0.1)^n = \frac{1-(0.1)^5}{1-(0.1)} = 1.1111$$
$$\sum_{n=0}^{3} (0.75)^n = \frac{1-(0.75)^4}{1-(0.75)} = 2.7344$$

3. *Binomial Expansion:*

$$(1+x)^n = \frac{1}{0!} + \frac{n}{1!}x + \frac{n\,(n-1)}{2!}x^2 + \frac{n\,(n-1)\,(n-2)}{3!}x^3 + \cdots + \frac{n!}{(n-r)!r!}x^r + \cdots$$
$$\equiv \left(\begin{array}{c} n \\ 0 \end{array}\right) + \left(\begin{array}{c} n \\ 1 \end{array}\right) x + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^2 + \left(\begin{array}{c} n \\ 3 \end{array}\right) x^3 + \cdots + \left(\begin{array}{c} n \\ r \end{array}\right) x^3 x^r + \cdots$$
$$= 1 + nx + \frac{n\,(n-1)}{2}x^2 + \frac{n\,(n-1)\,(n-2)}{6}x^3 + \cdots$$

$$\textit{where } \left(\begin{array}{c} n \\ r \end{array}\right) \equiv \frac{n!}{(n-r)!r!} \textit{ and } 0! \equiv 1$$

If n is a positive integer, the series includes $n+1$ terms. If n is NOT a positive integer, the series converges if $|x| < 1$. If $n > 0$, the series converges if $|x| = 1$.

$$\Longrightarrow\ (1-x)^n = 1 + n\,(-x) + \frac{n\,(n-1)}{2}\,(-x)^2 + \frac{n\,(n-1)\,(n-2)}{6}\,(-x)^3 + \cdots$$
$$= 1 - nx + \frac{n\,(n-1)}{2!}x^2 - \frac{n\,(n-1)\,(n-2)}{3!}x^3 + \cdots$$

Example:

$$(1+x)^{\frac{1}{2}} = \sqrt{(1+x)} = 1 + \frac{1}{2}\cdot x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\cdot x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\cdot x^2 + \cdots$$
$$= 1 + \frac{1}{2}\cdot x - \frac{1}{8}\cdot x^2 + \frac{1}{16}\cdot x^3 + \cdots$$

$$(1-x)^{\frac{1}{3}} = \sqrt[3]{(1-x)} = 1 + \frac{1}{3}\cdot x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\cdot x^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\cdot x^2 + \cdots$$
$$= 1 + \frac{1}{3}\cdot x - \frac{1}{9}\cdot x^2 + \frac{5}{81}\cdot x^3 + \cdots$$

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4. *Exponential*

$$\begin{aligned}\exp[u] = e^u &= \sum_{n=0}^{\infty} \frac{u^n}{n!} \\ &= \frac{1}{0!} + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \\ &= 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + \cdots\end{aligned}$$

5. *Complex Exponential*

$$\begin{aligned}\exp[+i\theta] = e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \frac{1}{0!} + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= 1 + i\theta + \frac{1}{2}i^2\theta^2 + \frac{1}{6}i^3\theta^3 + \frac{1}{24}i^4\theta^4 + \frac{1}{120}i^5\theta^5 + \cdots \\ &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right) + i \cdot \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)\end{aligned}$$

From Euler relation:

$$\exp[+i\theta] = \cos[\theta] + i\sin[\theta]$$

Equate real and imaginary parts:

$$\begin{aligned}\cos[\theta] &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots \\ \lim_{\theta\to 0} \cos[\theta] &= 1 \\ \sin[\theta] &= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots \\ \lim_{\theta\to 0} \sin[\theta] &= \theta\end{aligned}$$

6. *Maclaurin's Series:*

$$\begin{aligned} f\left[x\right] &= \frac{x^0}{0!} \cdot f\left[0\right] + \frac{x^1}{1!} \cdot \left.\frac{df}{dx}\right|_{x=0} + \frac{x^2}{2!} \cdot \left.\frac{d^2 f}{dx^2}\right|_{x=0} + \frac{x^3}{3!} \cdot \left.\frac{d^3 f}{dx^2}\right|_{x=0} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \cdot \left.\frac{d^n f}{dx^n}\right|_{x=0} \right) \\ &= f\left[0\right] + x \cdot f'\left[0\right] + \frac{x^2}{2} \cdot f''\left[0\right] + \frac{x^3}{6} \cdot f'''\left[0\right] + \cdots = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!} \cdot f^{(n)}\left[0\right] \right) \end{aligned}$$

7. *Taylor's Series:*

$$\begin{aligned} f\left[x + x_0\right] &= \frac{x_0^0}{0!} \cdot f\left[x\right] + \frac{x_0}{1!} \cdot \frac{df}{dx} + \frac{x_0^2}{2!} \cdot \frac{d^2 f}{dx^2} + \frac{x_0^3}{3!} \cdot \frac{d^3 f}{dx^2} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot \left.\frac{d^n f}{dx^n}\right|_{x=0} \right) \\ &= f\left[0\right] + x_0 \cdot f'\left[0\right] + \frac{x_0^2}{2} \cdot f''\left[0\right] + \frac{x_0^3}{6} \cdot f'''\left[0\right] + \cdots = \sum_{n=0}^{\infty} \left(\frac{x_0^n}{n!} \cdot f^{(n)}\left[0\right] \right) \end{aligned}$$