1. Find expressions for AND SKETCH the even and odd parts of $e^{i\cdot 2\pi \frac{x}{4}} = \exp \left[ +i \cdot 2\pi \frac{x}{4} \right]$. The x-axes of the sketches should be from at least $-8 \leq x \leq +8$.

**Definitions of even and odd parts**

\[
\begin{align*}
    f_{\text{even}}[x] &= \frac{f[x] + f[-x]}{2} \\
    f_{\text{odd}}[x] &= \frac{f[x] - f[-x]}{2}
\end{align*}
\]

so the function is the sum of its even and odd parts:

\[ f[x] = f_{\text{even}}[x] + f_{\text{odd}}[x] \]

In this case, we can use the Euler relation to decompose $f[x]$ into its real and imaginary parts:

\[ e^{+i\theta} = \cos[\theta] + i \cdot \sin[\theta] \]

\[ \Rightarrow f[x] = \exp \left[ +i \cdot 2\pi \frac{x}{4} \right] = \cos \left[ 2\pi \frac{x}{4} \right] + i \cdot \sin \left[ 2\pi \frac{x}{4} \right] \]

The reversed replica of the function is:

\[ f[-x] = \exp \left[ +i \cdot 2\pi \frac{(-x)}{4} \right] = \exp \left[ -i \cdot 2\pi \frac{x}{4} \right] \]

\[ = \cos \left[ 2\pi \left( -\frac{x}{4} \right) \right] + i \cdot \sin \left[ 2\pi \left( -\frac{x}{4} \right) \right] \]

\[ = \cos \left[ 2\pi \left( \frac{x}{4} \right) \right] - i \cdot \sin \left[ 2\pi \left( \frac{x}{4} \right) \right] \]

where the facts that the cosine is even and the sine is odd have been used. Therefore the even part is

\[
\begin{align*}
    f_{\text{even}}[x] &= \frac{\left( \exp \left[ +i \cdot 2\pi \frac{x}{4} \right] + \exp \left[ -i \cdot 2\pi \frac{x}{4} \right] \right)}{2} \\
    &= \frac{\left( \cos \left[ 2\pi \frac{x}{4} \right] + i \cdot \sin \left[ 2\pi \frac{x}{4} \right] \right) + \left( \cos \left[ 2\pi \frac{x}{4} \right] - i \cdot \sin \left[ 2\pi \frac{x}{4} \right] \right)}{2} \\
    &= \frac{2 \cdot \cos \left[ 2\pi \frac{x}{4} \right] + i \cdot 0[x]}{2} = \cos \left[ 2\pi \frac{x}{4} \right]
\end{align*}
\]

\[ f_{\text{even}}[x] = \cos \left[ 2\pi \frac{x}{4} \right] \]

Note that the even part is also real!
$f_{\text{odd}} [x] = \frac{(\exp [+i \cdot 2\pi \frac{x}{4}] - \exp [-i \cdot 2\pi \frac{x}{4}])}{2}
= \frac{(\cos [2\pi \frac{x}{4}] + i \cdot \sin [2\pi \frac{x}{4}]) - (\cos [2\pi \frac{x}{4}] - i \cdot \sin [2\pi \frac{x}{4}])}{2}
= \frac{2 \cdot i \cdot \sin [2\pi \frac{x}{4}]}{2}
= i \cdot \sin [2\pi \frac{x}{4}]

The odd part is also imaginary!

Side comment: many of you cancelled the factors of 2 in the argument of the exponential. Though this is clearly correct, I advise you NOT to do it because the form:

$\Phi [x] = 2\pi \frac{x}{4}$

is in the form that IMMEDIATELY tells you that the period is 4 units of length, and therefore that the spatial oscillation frequency is $\frac{1}{4}$ cycle per unit length. Several of you performed the cancellation and then did not plot the graph correctly, perhaps because the hint of the period was no longer present.
2. Find expressions for AND SKETCH the magnitude and the phase of \( e^{+i \cdot 2\pi \frac{x}{4}} \). The x-axes of the sketches should be from at least \(-8 \leq x \leq +8\).

*The function is already written as magnitude and phase!*

\[
\begin{align*}
f[x] &= \exp \left[ +i \cdot 2\pi \frac{x}{4} \right] = 1[x] \cdot \exp \left[ +i \cdot 2\pi \frac{x}{4} \right] = |f[x]| \cdot \exp [+i \cdot \Phi \{f[x]\}] \\
|f[x]| &= 1[x] \\
\Phi \{f[x]\} &= +2\pi \frac{x}{4}
\end{align*}
\]

so all you had to do was to plot these as FUNCTIONS OF x! (which some of you did not do ... some tried to plot the Argand diagram, though the problem gave a specific domain over which to plot these)

![Magnitude of f(x)](image1)

\[|f[x]| = \exp \left[ +i \cdot 2\pi \frac{x}{4} \right] = 1[x]\]

![Phase of f(x)](image2)

\[\Phi \{f[x]\} = +2\pi \frac{x}{4}\]
3. Find expressions for AND SKETCH the even and odd parts of $e^{+i\pi \frac{x^2}{4}} = \exp \left[ +i \cdot \pi \left( \frac{x}{2} \right)^2 \right]$.

The x-axes of the sketches should be from at least $-4 \leq x \leq +4$.

This is the “quadratic-phase function;” use the Euler relation to break up in real and imaginary parts:

$$e^{+i\theta} = \cos [\theta] + i \cdot \sin [\theta]$$

$$f [x] = \exp \left[ +i \cdot \pi \left( \frac{x}{2} \right)^2 \right]$$

$$= \cos \left[ \pi \left( \frac{x}{2} \right)^2 \right] + i \cdot \sin \left[ \pi \left( \frac{x}{2} \right)^2 \right]$$

You might then evaluate the “reversed” function:

$$f [-x] = \cos \left[ \pi \left( -\frac{x}{2} \right)^2 \right] + i \cdot \sin \left[ \pi \left( -\frac{x}{2} \right)^2 \right]$$

$$= \cos \left[ \pi \left( \frac{x}{2} \right)^2 \right] + i \cdot \sin \left[ \pi \left( \frac{x}{2} \right)^2 \right] = f [+x]$$

**BUT:** note that

$$\cos \left[ \pi \left( -\frac{x}{2} \right)^2 \right] = \cos \left[ \pi \left( \frac{x}{2} \right)^2 \right]$$

and:

$$\sin \left[ \pi \left( -\frac{x}{2} \right)^2 \right] = \sin \left[ \pi \left( \frac{x}{2} \right)^2 \right]$$

which means that BOTH the real part AND the imaginary part are EVEN. Several of you fell into the trap of thinking that because $\sin [-\theta] = -\sin [+\theta]$, therefore ALL sinusoidal functions are odd, even if the argument is even. This means that $f [x] = \exp \left[ +i \cdot \pi \left( \frac{x}{2} \right)^2 \right]$ IS EVEN (The odd part is $0 [x]$)

$$f_{\text{even}} [x] = \cos \left[ \pi \left( \frac{x}{2} \right)^2 \right] + i \cdot \sin \left[ \pi \left( \frac{x}{2} \right)^2 \right]$$

So the even part has both real and imaginary parts:
and the odd part is zero for all $x$:

$$\text{Odd part of } f[x] = 0[x] + i \cdot 0[x] = 0[x]$$
4. Find expressions for AND SKETCH the magnitude and the phase of $e^{+ix^2} = \exp\left[+i \cdot \pi \left(\frac{x^2}{2}\right)^2\right]$. The x-axes of the sketches should be from at least $-4 \leq x \leq +4$.

Again, this is already in the form of magnitude and phase:

$$f[x] = \exp\left[+i \cdot \pi \left(\frac{x}{2}\right)^2\right] = 1[x] \cdot \exp\left[+i \cdot \pi \left(\frac{x^2}{2}\right)\right]$$

| $|f[x]|$ | 1.0 |
|--------|-----|
| 0.8    |     |
| 0.6    |     |
| 0.4    |     |
| 0.2    |     |

$|f[x]| = \exp\left[+i \cdot 2\pi \frac{x^2}{4}\right] = 1[x]$  

Phase of $f[x]$ (radians)  

$\Phi\{f[x]\} = +2\pi \frac{x}{4}$
5. Find the projection of the vector \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ +1 \end{bmatrix} \) onto the reference vector \( \mathbf{a}_0 = \begin{bmatrix} 1 \\ e^{-i\pi/4} \\ -i \end{bmatrix} \)

(HINT: be careful!)

First, many of you did not remember that we use the convention that the reference vector is listed FIRST; if you don’t follow this convention, then you will get an incorrect result in cases where the both vectors have complex components – this is the first of the “be careful” parts.

projection of \( \mathbf{v}_1 \) onto \( \mathbf{a}_0 \) = \( \frac{\hat{\mathbf{a}}_0 \cdot \mathbf{v}_1}{|\mathbf{a}_0|} \) (this is the other of the “be careful” parts)

Length of reference \( |\mathbf{a}_0| = \sqrt{(1^2 + |e^{-i\pi/4}|^2 + |-i|^2)} = \sqrt{1 + 1 + 1} = \sqrt{3} \)

\( \mathbf{a}_0 \cdot \mathbf{v}_1 = \sum_{n=0}^{1} (\mathbf{a}_0)_n^* \times (\mathbf{v}_1)_n \)

\( = (\mathbf{a}_0)_0^* \times (\mathbf{v}_1)_0 + (\mathbf{a}_0)_1^* \times (\mathbf{v}_1)_1 + (\mathbf{a}_0)_2^* \times (\mathbf{v}_1)_2 \)

\( = \frac{1}{\sqrt{3}} \cdot \left( 1^* \cdot 1 + \left( \exp \left[ -i \frac{\pi}{4} \right]^* \cdot (-1) \right) + ((-i)^* \cdot (+1)) \right) \)

\( = \frac{1}{\sqrt{3}} \cdot \left( 1 - \exp \left[ +i \frac{\pi}{4} \right] + i \right) = \frac{1}{\sqrt{3}} \cdot \left( 1 + i - \exp \left[ +i \frac{\pi}{4} \right] \right) \)

Recall that:

\( \exp \left[ +i \frac{\pi}{4} \right] = \cos \left[ \frac{\pi}{4} \right] + i \cdot \sin \left[ \frac{\pi}{4} \right] \)

\( = \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \)

Projection of \( \mathbf{v}_1 \) onto \( \mathbf{a}_0 \) is:

\( \frac{1}{\sqrt{3}} \cdot \left( 1 + i - \exp \left[ +i \frac{\pi}{4} \right] \right) = \frac{1}{\sqrt{3}} \cdot \left( 1 + i - \left( \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \right) \right) \)

\( = \frac{1}{\sqrt{3}} \cdot \left( \left( 1 - \frac{1}{\sqrt{2}} \right) - i \cdot \left( 1 - \frac{1}{\sqrt{2}} \right) \right) \)

Just in case you evaluated it (you did not have to), the magnitude of the projection is:

\( \frac{1}{\sqrt{3}} \cdot \sqrt{\left( \left( 1 - \frac{1}{\sqrt{2}} \right)^2 + \left( 1 - \frac{1}{\sqrt{2}} \right)^2 \right)} = \frac{1}{\sqrt{3}} \cdot \sqrt{2 \cdot \left( 1 - \frac{1}{\sqrt{2}} \right)^2} \)

\( = \sqrt{\frac{2}{3}} \cdot \left( 1 - \frac{1}{\sqrt{2}} \right) \)

\( = \sqrt{\frac{2}{3}} \cdot \left( \frac{\sqrt{2} - 1}{\sqrt{2}} \right) \)

\( = \frac{\sqrt{2} - 1}{\sqrt{3}} \)
and the phase angle is:

\[
\phi = \tan^{-1} \left[ \frac{\frac{1}{\sqrt{3}} \cdot \left(1 - \frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{3}} \cdot \left(1 - \frac{1}{\sqrt{2}}\right)} \right] = \tan^{-1}[1] = +\frac{\pi}{4}
\]
6. Evaluate the projections of the 4-D input vector \( \mathbf{x} = \begin{bmatrix} -2 \\ i \\ 2 \\ -i \end{bmatrix} \) onto each of the 4 eigenvectors of a 4 × 4 shift-invariant matrix.

Since the matrix is circulant (shift invariant), then we already know the four eigenvectors:

\[
\hat{\mathbf{x}}_0 = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{\mathbf{x}}_1 = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix}, \quad \hat{\mathbf{x}}_2 = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \hat{\mathbf{x}}_3 = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ -i \\ 1 \\ +i \end{bmatrix}
\]

Again, when projecting the test vector onto the reference, you MUST evaluate the complex conjugates of the components of the reference vector:

\[
\frac{1}{2} \cdot 1 \cdot \begin{bmatrix} +1 \\ +1 \\ +1 \\ +1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ i \\ 2 \\ -i \end{bmatrix} = \frac{1}{2} \cdot (1 \cdot (-2) + 1 \cdot (+i) + 1 \cdot (+i) + 1 \cdot (-i))
\]

so the input vector is orthogonal to the constant vector. The projection onto the vector with frequency \( \xi = \frac{1}{4} \) cycle per sample is:

\[
\frac{1}{2} \cdot \begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix} \cdot \begin{bmatrix} -2 \\ i \\ 2 \\ -i \end{bmatrix} = \frac{1}{2} \cdot (1 \cdot (-2) + (+i) \cdot (+i) + (-1) \cdot 2 + (-i) \cdot (-i))
\]

\[
= \frac{1}{2} \cdot ((-2) + (+i) \cdot (+i) + (-1) \cdot 2 + (+i) \cdot (-i))
\]

\[
= \frac{1}{2} \cdot (-2 + 1 - 1) = -1
\]

The input vector is orthogonal to the vector with frequency \( \frac{1}{2} \) cycle per sample:

\[
\frac{1}{2} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ i \\ 2 \\ -i \end{bmatrix} = \frac{1}{2} \cdot (-2 - i + 2 + i) = 0
\]

and the projection onto the vector with frequency \( \xi = \frac{3}{4} \) cycle per sample is

\[
\frac{1}{2} \cdot \begin{bmatrix} +1 \\ -i \\ -1 \\ +i \end{bmatrix} \cdot \begin{bmatrix} -2 \\ i \\ 2 \\ -i \end{bmatrix} = \frac{1}{2} \cdot (1 \cdot (-2) + (-i) \cdot (+i) + (-1) \cdot 2 + (+i) \cdot (-i))
\]

\[
= \frac{1}{2} \cdot [(-2) + i^2 - 2 + i^2] = \frac{1}{2} \cdot [-2 - 1 - 2 - 1] = \frac{1}{2} \cdot (-6) = -3
\]
The input vector may be written as a weighted sum of the four “pixel vectors:”

\[
\mathbf{x} = \begin{bmatrix} -2 \\ \frac{i}{2} \\ -i \end{bmatrix} = (-2) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-i) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

or as a weighted sum of the four eigenvectors:

\[
\mathbf{x} = 0 \cdot \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) + (-1) \cdot \left( \begin{bmatrix} 1 \\ \frac{-i}{2} \\ -1 \\ -i \end{bmatrix} \right) + 0 \cdot \left( \begin{bmatrix} 1 \\ \frac{-1}{2} \\ 1 \\ -1 \end{bmatrix} \right) + (-3) \cdot \left( \begin{bmatrix} 1 \\ \frac{-i}{2} \\ -1 \\ +i \end{bmatrix} \right)
\]

\[
= -\frac{1}{2} \begin{bmatrix} 1 \\ +i \\ -1 \\ -i \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ -i \\ -1 \\ +i \end{bmatrix} = \begin{bmatrix} -2 \\ +i \\ +2 \\ -i \end{bmatrix}
\]

Note that you can also find the projections by applying the matrix of complex-conjugates of eigenvectors to the input vector:

\[
\mathbf{D}^{-1} \mathbf{x} = \mathbf{x}'
\]

\[
\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix} \begin{bmatrix} -2 \\ +i \\ +2 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -3 \end{bmatrix}
\]
7. By any method (but state how) determine the four eigenvalues of the $4 \times 4$ circulant matrix listed and use to determine if $\mathbf{A}^{-1}$ exists.

$$
\mathbf{A} = \begin{bmatrix}
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1
\end{bmatrix}
$$

This is a circulant matrix, so we know the 4 eigenvectors. We can evaluate the output of this matrix for each of those 4 vectors to determine the scale factor:

$$
\mathbf{A} \hat{\mathbf{x}}_0 = \lambda_0 \hat{\mathbf{x}}_0
$$

$$
\begin{bmatrix}
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1
\end{bmatrix}
\begin{bmatrix}
+\frac{1}{2} \\
+\frac{1}{2} \\
-\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \mathbf{0} 
\implies \lambda_0 = 0
$$

$$
\mathbf{A} \hat{\mathbf{x}}_1 = \lambda_1 \hat{\mathbf{x}}_1
$$

$$
\begin{bmatrix}
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1
\end{bmatrix}
\begin{bmatrix}
1 + i \\
1 + i \\
1 - i \\
1 - i
\end{bmatrix}
= (2 + 2i)
\begin{bmatrix}
+\frac{1}{2} \\
+\frac{1}{2} \\
-\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
\implies \lambda_1 = 2 + 2i
$$

$$
\mathbf{A} \hat{\mathbf{x}}_2 = \lambda_0 \hat{\mathbf{x}}_2
$$

$$
\begin{bmatrix}
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
= \mathbf{0} 
\implies \lambda_2 = 0
$$

$$
\mathbf{A} \hat{\mathbf{x}}_3 = \lambda_3 \hat{\mathbf{x}}_3
$$

$$
\begin{bmatrix}
+1 & +1 & -1 & -1 \\
-1 & +1 & +1 & -1 \\
-1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1
\end{bmatrix}
\begin{bmatrix}
1 - i \\
1 - i \\
1 + i \\
1 + i
\end{bmatrix}
= (2 - 2i)
\begin{bmatrix}
+\frac{1}{2} \\
+\frac{1}{2} \\
-\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
\implies \lambda_3 = 2 - 2i
$$

So $\lambda_0 = \lambda_2 = 0 \implies$ those two eigenvectors are "blocked" $\implies \mathbf{A}^{-1}$ does not exist