1051-320 Homework #3 Due 1/11/2007 (Th)

(for these problems, you may use a computer program, such as Mathematica or MatLab, to get some confirmation, but I am looking for reasoning for most of the answers)

1. For the following matrices:

$$\underline{\mathbf{A}}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \underline{\mathbf{A}}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \underline{\mathbf{A}}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \underline{\mathbf{A}}_4 = \begin{pmatrix} +\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & +\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & +\frac{1}{2} \end{pmatrix}$$

(a) Write down expressions for the three components that define the vectors in the *row subspace* and in the *null subspace* of each matrix

(b) Show that the vector(s) in the null subspace is orthogonal to the vector(s) in the row subspace.

(c) Do the inverses of these matrices exist? Explain why or why not.

(d) (OPTIONAL BONUS) Find the inverses $\underline{\mathbf{A}}_n^{-1}$ if they exist.

(e) (OPTIONAL BONUS) In cases where the inverse matrix matrix does not exist, find the matrix that does the "best job" of inverting the problem; this is the *pseudoinverse*.

2. A 3×3 sampled "object" with real-valued gray values has the form $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix}$ (I did not use "i" for a reason) It also may be represented as a 9-D vector by "stacking" the columns to obtain

$$\underline{\mathbf{x}} \equiv \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ j \end{bmatrix}$$

(this is the "lexicographically" ordered 1-D vector that represents the 2-D image). In each case listed below, find the matrix operators $\underline{\mathbf{A}}_n$ that produce the following output vectors $\underline{\mathbf{b}}_n$ ("output images") when applied to $\underline{\mathbf{x}}$:

(a) $\underline{\mathbf{b}}_1$ is vector for the original image after exchanging rows and columns, so that the 3×3 output image is $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$.

(b) $\underline{\mathbf{b}}_2$ is vector corresponding to the original image rotated by $+90°$

(c) $\underline{\mathbf{b}}_3$ is the original image rotated by $+180°$

(d) $\underline{\mathbf{b}}_4$ is vector corresponding to the original image rotated by $+270°$

MORE→→

(e) The elements of the output image are the sums of the rows, of the columns, and of an approximation of the diagonals of $\underline{\mathbf{x}}$;

$$\underline{\mathbf{b}}_5 = \begin{bmatrix} a+d+g \\ b+e+h \\ c+f+j \\ a+b+c \\ d+e+f \\ g+h+j \\ b+c+f \\ a+e+j \\ d+g+h \end{bmatrix}$$

This is not just a theoretical example, but a very simplified version of a real imaging problem. In many applications, and particularly in medical imaging, we need to reconstruct the original input object $\underline{\mathbf{x}}$ from the sums (line-integral projections) of the image data along paths at various angles through the object. The mathematics are used in several medical imaging modalities (*e.g.,* computed tomography and magnetic resonance imaging), as well as other imaging applications.

(f) (OPTIONAL BONUS) Find the inverses $\underline{\mathbf{A}}_n^{-1}$ of the matrix operators *if they exist.* If $\underline{\mathbf{A}}_n^{-1}$ does not exist, then explain why. (*n.b.,* despite appearances, there is an easy way to see this for $\underline{\mathbf{b}}_5$