1 General 2 × 2 circulant matrix

$$\underline{\mathbf{A}} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Eigenvectors:

$$\begin{aligned}
\hat{\underline{\mathbf{x}}}_0 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\hat{\underline{\mathbf{x}}}_1 &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

Eigenvalues obtained by applying matrix to known eigenvectors:

$$\begin{aligned}
\underline{\mathbf{A}}\hat{\underline{\mathbf{x}}}_0 &= \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = (a+b) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Longrightarrow \lambda_0 = a+b \\
\underline{\mathbf{A}}\hat{\underline{\mathbf{x}}}_1 &= \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = (a-b) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Longrightarrow \lambda_1 = a-b
\end{aligned}$$

Diagonalization matrix formed from eigenvectors IN ORDER

$$\begin{aligned}
\underline{\mathbf{D}}_2 &= [[\hat{\underline{\mathbf{x}}}_0]\,[\hat{\underline{\mathbf{x}}}_1]] = \left[\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
\underline{\mathbf{D}}_2^{-1} &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \underline{\mathbf{D}}_2^* \\
\underline{\mathbf{D}}_2^{-1}\underline{\mathbf{A}}\underline{\mathbf{D}}_2 &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \\
&= \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \equiv \underline{\mathbf{\Lambda}}
\end{aligned}$$

$$\begin{aligned}
\lambda_0 &= a+b \\
\lambda_1 &= a-b
\end{aligned}$$

2 General 3 × 3 circulant matrix formed from the three eigenvectors in order

$$\underline{\mathbf{D}}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp\left[+i\frac{2\pi}{3}\right] & \exp\left[+i\frac{4\pi}{3}\right] \\ 1 & \exp\left[+i\frac{4\pi}{3}\right] & \exp\left[+i\frac{8\pi}{3}\right] \end{bmatrix}$$

$$\underline{\mathbf{D}}_3^{-1} = \underline{\mathbf{D}}_3^{*} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp\left[-i\frac{2\pi}{3}\right] & \exp\left[-i\frac{4\pi}{3}\right] \\ 1 & \exp\left[-i\frac{4\pi}{3}\right] & \exp\left[-i\frac{8\pi}{3}\right] \end{bmatrix}$$

$$\underline{\mathbf{D}}_3^{-1} A \underline{\mathbf{D}}_3$$

$$= \left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp\left[-i\frac{2\pi}{3}\right] & \exp\left[-i\frac{4\pi}{3}\right] \\ 1 & \exp\left[-i\frac{4\pi}{3}\right] & \exp\left[-i\frac{8\pi}{3}\right] \end{bmatrix}\right)\left(\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\right)\left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \exp\left[+i\frac{2\pi}{3}\right] & \exp\left[+i\frac{4\pi}{3}\right] \\ 1 & \exp\left[+i\frac{4\pi}{3}\right] & \exp\left[+i\frac{8\pi}{3}\right] \end{bmatrix}\right)$$

$$= \begin{bmatrix} (a+b+c) & 0 & 0 \\ 0 & \left(a + b\exp\left[+i\frac{2\pi}{3}\right] + c\exp\left[+i\frac{4\pi}{3}\right]\right) & 0 \\ 0 & 0 & \left(a + b\exp\left[+i\frac{4\pi}{3}\right] + c\exp\left[+i\frac{8\pi}{3}\right]\right) \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \equiv \underline{\mathbf{\Lambda}}$$

$$\lambda_0 = a + b + c$$

$$\lambda_1 = a + b\exp\left[+i\frac{2\pi}{3}\right] + c\exp\left[+i\frac{4\pi}{3}\right]$$

$$\lambda_2 = a + b\exp\left[+i\frac{4\pi}{3}\right] + c\exp\left[+i\frac{8\pi}{3}\right] = a + b\exp\left[-i\frac{2\pi}{3}\right] + c\exp\left[-i\frac{4\pi}{3}\right]$$

3 General 4 × 4 circulant matrix formed from the four eigenvectors in order

$$\underline{\mathbf{D}}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$\underline{\mathbf{D}}_4^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\underline{\mathbf{D}}_4 \cdot \underline{\mathbf{D}}_4^{-1} = \underline{\mathbf{D}}_4^{-1}\underline{\mathbf{D}}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \underline{\mathbf{I}}$$

$$\underline{\mathbf{A}} = \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

$$\underline{\mathbf{D}}_4^{-1}\underline{\mathbf{A}}\underline{\mathbf{D}}_4 = \begin{bmatrix} a+b+c+d & 0 & 0 & 0 \\ 0 & a+ib-c-id & 0 & 0 \\ 0 & 0 & a-b+c-d & 0 \\ 0 & 0 & 0 & a-ib-c+id \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \equiv \underline{\mathbf{\Lambda}}$$

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