1 1051-320 Supplemental Notes, 12/21/2006

1.1 Transpose Matrix and Pseudoinverse Matrix

In an inverse problem, a "nonsquare" matrix with more rows than columns yields more "equations" than unknowns. Consider the matrix $\underline{\mathbf{A}}$ with 3 rows and 2 columns:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix}$$

This matrix may be applied to a two-element vector to produce a three-element vector, *e.g.,*

$$\begin{aligned} \underline{\mathbf{x}} &= \begin{bmatrix} 3 \\ -17 \end{bmatrix} \\ \underline{\mathbf{A}}\,\underline{\mathbf{x}} &= \underline{\mathbf{b}} \\ \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -17 \end{bmatrix} &= \begin{bmatrix} 3 \\ -62 \\ -56 \end{bmatrix} \end{aligned}$$

Now, what if $\underline{\mathbf{A}}$ and $\underline{\mathbf{b}}$ are known and $\underline{\mathbf{x}}$ is to be found; we have three equations in two unknowns:

$$\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ -62 \\ -56 \end{bmatrix}$$

which implies that:

$$\begin{aligned} 1 \cdot x_0 + 0 \cdot x_1 &= 3 \\ 2 \cdot x_0 + 4 \cdot x_1 &= -62 \\ 4 \cdot x_0 + 4 \cdot x_1 &= -56 \end{aligned}$$

We could use the principles of algebra to solve these three simultaneous equations, but let's try to use matrices. We know the system matrix $\underline{\mathbf{A}}$ (3 rows and 2 columns) and the 3-element vector $\underline{\mathbf{b}}$. We need to construct a matrix from $\underline{\mathbf{A}}$ that can operate legitimately on $\underline{\mathbf{b}}$. Our only choice is the transpose of $\underline{\mathbf{A}}$, which is obtained by "swapping" the rows and columns:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix} \Longrightarrow \underline{\mathbf{A}}^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

We can apply this from the left to both sides of the matrix-vector product:

$$\begin{aligned} \underline{\mathbf{A}}^T \left(\underline{\mathbf{A}}\,\underline{\mathbf{x}}\right) &= \underline{\mathbf{A}}^T \underline{\mathbf{b}} \\ \left(\underline{\mathbf{A}}^T \underline{\mathbf{A}}\right) \underline{\mathbf{x}} &= \underline{\mathbf{A}}^T \underline{\mathbf{b}} \end{aligned}$$

The product $\underline{\mathbf{A}}^T\underline{\mathbf{A}}$ is square; in this case, $\underline{\mathbf{A}}^T\underline{\mathbf{A}}$ is 2×2:

$$\underline{\mathbf{A}}^T\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 21 & 24 \\ 24 & 32 \end{bmatrix}$$

The determinant of this square matrix is nonzero:

$$\det \begin{bmatrix} 21 & 24 \\ 24 & 32 \end{bmatrix} = 21 \cdot 32 - 24 \cdot 24 = 96$$

which means that its inverse exists. We can use the equation for the inverse of a 2×2 matrix that we derived last class to find the inverse:

$$\begin{aligned} \begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \\ \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} &= \frac{1}{96} \begin{bmatrix} 32 & -24 \\ -24 & 21 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{7}{32} \end{bmatrix} \end{aligned}$$

We can multiply both sides of the equation by this term:

$$\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right) \underline{\mathbf{x}} = \left(\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \underline{\mathbf{A}}^T\right) \underline{\mathbf{b}}$$

But

$$\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right) = \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{7}{32} \end{bmatrix} \cdot \begin{bmatrix} 21 & 24 \\ 24 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\mathbf{I}}$$

So the product of all the terms yields:

$$\begin{aligned} \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right) \underline{\mathbf{x}} &= \underline{\mathbf{I}}\,\underline{\mathbf{x}} = \underline{\mathbf{x}} \\ &= \left(\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \underline{\mathbf{A}}^T\right) \underline{\mathbf{b}} \end{aligned}$$

or:

$$\boxed{\underline{\mathbf{x}} = \left(\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \underline{\mathbf{A}}^T\right) \underline{\mathbf{b}}}$$

This equation produces the matrix that "inverts" the imaging problem, but it is not the strict "inverse matrix." We call it the *Moore-Penrose Pseudoinverse* and label it by a superscript "dagger:"

$$\left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \underline{\mathbf{A}}^T \equiv \underline{\mathbf{A}}^\dagger$$

In the current problem, the pseudoinverse is:

$$\begin{aligned} \underline{\mathbf{A}}^\dagger &= \left(\underline{\mathbf{A}}^T\underline{\mathbf{A}}\right)^{-1} \cdot \underline{\mathbf{A}}^T = \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{7}{32} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \end{bmatrix} \end{aligned}$$

Check the result by applying this matrix to the "known" vector $\underline{\mathbf{b}}$

$$\underline{\mathbf{A}}^{\dagger}\underline{\mathbf{b}} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} 3 \\ -62 \\ -56 \end{bmatrix} = \begin{bmatrix} 3 \\ -17 \end{bmatrix} = \underline{\mathbf{x}}$$

which is the original input vector.

The Moore-Penrose pseudoinverse appears in many contexts, and not just in imaging. If errors exist in the measurements of the known "output vector" $\underline{\mathbf{b}}$, the application of $\underline{\mathbf{A}}^{\dagger}$ produces the "optimum estimate" of the original input vector $\underline{\mathbf{x}}$ in a least-squares sense, *i.e.*, the sum of the squares of the errors of each component between $\underline{\mathbf{x}}$ and its estimate $\underline{\mathbf{A}}^{\dagger}\underline{\mathbf{b}}$ is a minimum.

1.2 Matrices as Imaging Systems

The input vector is viewed as samples of a 1-D continuous function $f[x]$, as in Figure 3.1 in the notes. The matrix that produces the ideal image is the identity:

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}} &= \underline{\mathbf{b}} \\ \underline{\mathbf{b}} &= \underline{\mathbf{x}} \text{ if } \underline{\mathbf{A}} = \underline{\mathbf{I}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

The "imaging system" that translates the input by one component (by one "sample" or one "pixel") to the right is

$$\underline{\mathbf{A}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Note that each row is derived from that before by a "circular translation" (the sample that "disappears" off the edge "reappears" on the other edge). This is a *circulant matrix*, and only contains N unique components.

The sum of the identity matrix and the translation matrix is:

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

This matrix computes a new image that is the sum of the original and an image translated one pixel to the right. The matrix that computes the average of those two pixels is:

$$\underline{\mathbf{A}} = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

In the 3×3 case, the determinant of the averaging matrix is:

$$\det \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{4}$$

which indicates that its inverse exists:

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Note that the determinant of the inverse matrix is the reciprocal of the determinant of the averaging matrix:

$$\det \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = 4$$

Also note that the inverse of the averaging matrix computes differences of the values; it is a "differencer" whereas the original matrix is an "averager" or an "integrator."

The 4×4 version of the 2-pixel averager is:

$$\underline{\mathbf{A}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Its determinant is:

$$\det \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 0$$

So this matrix is NOT invertible.

As a last example, consider the operator that subtracts the original image from the translated image. The matrix is:

$$\underline{\mathbf{A}} = \begin{bmatrix} -1 & 0 & \cdots & 0 & +1 \\ +1 & -1 & \cdots & 0 & 0 \\ 0 & +1 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & +1 & -1 \end{bmatrix}$$

In the 3×3 case, the matrix is:

$$\underline{\mathbf{A}} = \begin{bmatrix} -1 & 0 & +1 \\ +1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} -1 & 0 & +1 \\ +1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix} = 0$$

So the inverse does not exist. The 4×4 matrix is:

$$\underline{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 & +1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} -1 & 0 & 0 & +1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{bmatrix} = 0$$

So these matrices cannot be inverted. To see why, apply them to "constant input vectors:"

$$\begin{bmatrix} -1 & 0 & +1 \\ +1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -10+0+10 \\ +10-10+0 \\ 0+10-10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \equiv \underline{\mathbf{0}}$$

$$\begin{bmatrix} -1 & 0 & +1 \\ +1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix} = \begin{bmatrix} -20+0+20 \\ +20-20+0 \\ 0+20-20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \equiv \underline{\mathbf{0}}$$

Since two **distinct input vectors produce the same output, we cannot distinguish the two inputs from knowledge of the matrix $\underline{\mathbf{A}}$ the output vector $\underline{\mathbf{b}}$; we need some additional information** (what often is called a "boundary condition").