1 Supplemental Notes for 12/19/2006

1.1 2×2 Matrix

Given a 2×2 matrix $\underline{\mathbf{A}} \equiv \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$

$$\det \underline{\mathbf{A}} = \det \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \equiv a_{00}a_{11} - a_{01}a_{10}$$

In class, we derived the inverse of this 2×2 matrix $\underline{\mathbf{A}}$:

$$\underline{\mathbf{A}}^{-1} = \frac{1}{a_{00}a_{11} - a_{01}a_{10}} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} = \frac{1}{\det \underline{\mathbf{A}}} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix}$$

(n.b., swap the diagonal components and negate the off-diagonals).
Confirm that this works:

$$\underline{\mathbf{A}\mathbf{A}}^{-1} = \left(\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \right) \cdot \left(\frac{1}{a_{00}a_{11} - a_{01}a_{10}} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\mathbf{A}}^{-1}\underline{\mathbf{A}} = \left(\frac{1}{a_{00}a_{11} - a_{01}a_{10}} \begin{bmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that the inverse matrix exists ONLY if:

$$\det \underline{\mathbf{A}} = a_{00}a_{11} - a_{01}a_{10} \neq 0$$

For a matrix $\underline{\mathbf{A}}$ with equal column vectors:

$$\det \begin{bmatrix} a_{00} & a_{00} \\ a_{10} & a_{10} \end{bmatrix} = a_{00}a_{10} - a_{10}a_{00} = 0$$

So a 2×2 matrix with equal columns cannot be inverted. If the column vectors differ by some constant (say β):

$$\det \begin{bmatrix} a_{00} & \beta a_{00} \\ a_{10} & \beta a_{10} \end{bmatrix} = a_{00}\left(\beta a_{10}\right) - a_{10}\left(\beta a_{00}\right) = 0$$

Similarly for a matrix $\underline{\mathbf{A}}$ with equal row vectors:

$$\begin{aligned} \det \begin{bmatrix} a_{00} & a_{01} \\ a_{00} & a_{01} \end{bmatrix} &= 0 \\ \det \begin{bmatrix} a_{00} & a_{01} \\ \beta a_{00} & \beta a_{01} \end{bmatrix} &= 0 \end{aligned}$$

We can understand this by considering the imaging equation:

$$\begin{aligned} \underline{\mathbf{A}\mathbf{x}} &= \underline{\mathbf{b}} \\ &= \begin{bmatrix} a_{00} & a_{01} \\ \beta a_{00} & \beta a_{01} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ &\Longrightarrow a_{00}x_0 + a_{01}x_1 = b_0 \\ \text{and } \left(\beta a_{00}\right)x_0 + \left(\beta a_{01}\right)x_1 &= b_1 = \beta\left(a_{00}x_0 + a_{01}x_1\right) = \beta b_0 \end{aligned}$$

In words, these are really the same two equations, which means that the multiplication of $\underline{\mathbf{x}}$ by the second row vector of $\underline{\mathbf{A}}$ resulted in no additional information.

1.2 3×3 Matrix

$$\underline{\mathbf{A}} \equiv \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} \det \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} &= a_{00}a_{11}a_{22} - a_{00}a_{12}a_{21} - a_{01}a_{10}a_{22} + a_{01}a_{20}a_{12} + a_{10}a_{02}a_{21} - a_{02}a_{11}a_{20} \\ &= a_{00}\left(a_{11}a_{22} - a_{12}a_{21}\right) - a_{01}\left(a_{10}a_{22} - a_{20}a_{12}\right) + a_{02}\left(a_{10}a_{21} - a_{11}a_{20}\right) \end{aligned}$$

Computed Inverse Matrix $\underline{\mathbf{A}}^{-1}$ is a COMPLICATED expression

$$\begin{aligned} \underline{\mathbf{A}}^{-1} &= \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}^{-1} \\ &= \frac{1}{\det \underline{\mathbf{A}}} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{01}a_{22} - a_{02}a_{21} & a_{01}a_{12} - a_{02}a_{11} \\ a_{10}a_{22} - a_{20}a_{12} & a_{00}a_{22} - a_{02}a_{20} & a_{00}a_{12} - a_{10}a_{02} \\ a_{10}a_{21} - a_{11}a_{20} & a_{00}a_{21} - a_{01}a_{20} & a_{00}a_{11} - a_{01}a_{10} \end{bmatrix} \end{aligned}$$

But, what if the matrix $\underline{\mathbf{A}}$ is diagonal? Then the inverse is trivial to evaluate:

$$\begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_{00}} & 0 & 0 \\ 0 & \frac{1}{a_{11}} & 0 \\ 0 & 0 & \frac{1}{a_{22}} \end{bmatrix}$$

If one of the diagonal elements is zero, then the inverse does not exist:

$$\begin{bmatrix} a_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{22} \end{bmatrix}^{-1} = \mathit{undecidable}$$

But we may still "invert" the nonzero diagonal elements:

$$\underline{\mathbf{A}}^{-1} = \begin{bmatrix} a_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{22} \end{bmatrix}^{-1} \tilde{} \begin{bmatrix} \frac{1}{a_{00}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{22}} \end{bmatrix} \cong \underline{\mathbf{A}}^{\dagger}$$

where the matrix $\underline{\mathbf{A}}^{\dagger}$ ("A-dagger") is the *pseudoinverse* of $\underline{\mathbf{A}}$.