1 Subspaces Associated with a Matrix $\underline{\mathbf{A}}$

Consider the matrix $\underline{\mathbf{A}}$

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 4 \end{bmatrix}$$

that acts on a 2-D vector, e.g., $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ to produce a 3-D output vector $\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$.

1.1 Row Subspace

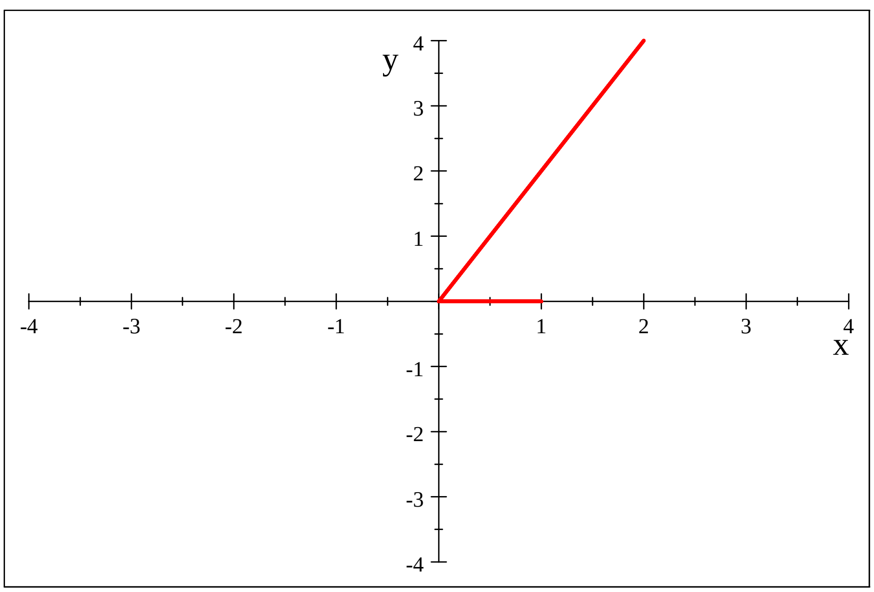
The set of vectors in the *row subspace* are the linear combinations (weighted sums) of the row vectors:

$$\{\underline{\mathbf{x}}_r\} = \alpha_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

BUT: the third row is a weighted sum of the first two:

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

which says that the third row vector does not provide any "new" (nonredundant) information that we could get from the first two. The set of vectors in the 2-D row subspace are those we can "get to" by adding weighted sums of these two:



The two 2-D vectors that define the row subspace of the matrix $\underline{\mathbf{A}}$.

which clearly includes all 2-D vectors.

1.2 Null Subspace

The set of vectors that are perpendicular to the vectors in the row subspace constitute the *null subspace*. In this example, it is the set of 2-D vectors that are simultaneously perpendicular to these two vectors:

$$0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot x_0 + 0 \cdot x_1 = x_0 \implies x_0 = 0$$

$$0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \cdot x_0 + 4 \cdot x_1 = 2 \cdot 0 + 4 \cdot x_1 \implies x_1 = 0$$

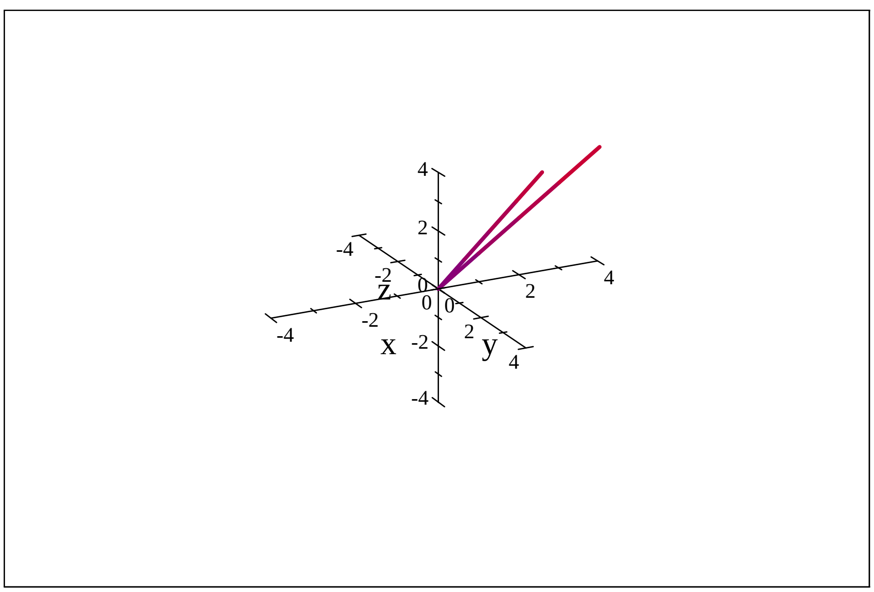
Thus the null subspace consists of only one vector:

$$\{\underline{\mathbf{x}}_n\} = \underline{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1.3 Column Subspace

The set of vectors that are weighted sums of the two 3-D column vectors constitute the *column subspace*, which is the set of output vectors $\{\underline{\mathbf{b}}_c\}$ that "we can get to" by applying the matrix $\underline{\mathbf{A}}$ to a vector in the row subspace $\{\underline{\mathbf{x}}\}$:

$$\{\underline{\mathbf{b}}_c\} = \gamma_0 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \gamma_1 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$



The two 3-D vectors that define the column subspace of the matrix $\underline{\mathbf{A}}$.

1.4 Left-Null Subspace

Since there are only two vectors that define the column subspace (and since these two vectors are not multiples of each other – the column subspace would be a 1-D line if they were), then nonnull vectors exist that are perpendicular to these two; these constitute the *left-null subspace:*

$$
\begin{aligned}
0 &= \begin{bmatrix} b_0 \\ b_1 \\ b_3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = b_0 + 2b_1 + 4b_2 \\
0 &= \begin{bmatrix} b_0 \\ b_1 \\ b_3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = 4b_1 + 4b_2 \implies b_2 = -b_1
\end{aligned}
$$

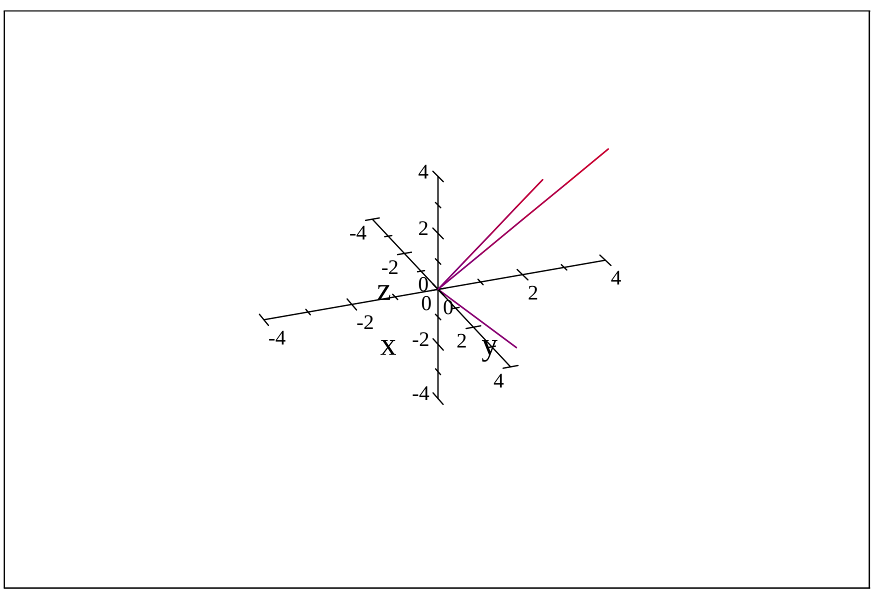
Substitute $b_2 = -b_1$ into the first equation:

$$0 = b_0 + 2b_1 - 4b_1 = b_0 - 2b_1 \implies b_0 = +2b_1$$

So the set of vectors that satisfy these two constraints is:

$$\{\underline{\mathbf{x}}_n\} = \gamma \begin{bmatrix} +2 \\ +1 \\ -1 \end{bmatrix}$$

which is easy to show is orthogonal to the two vectors that define the column subspace by evaluating the scalar product.



The vector that defines the left-null subspace is simultaneously perpendicular to the two vectors that define the column subspace.

1.5 Matrix Operator that Adds Rows and Columns of an "Image"

Consider the 2×2 discrete "image"

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We can convert this to a vector by "lexicographic ordering" (stack the column vectors):

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

The matrix operator $\underline{\mathbf{A}}$ that adds up the rows and columns of this "image" is 4×4:

$$\begin{aligned} \underline{\mathbf{A}}\underline{\mathbf{x}} &= \underline{\mathbf{b}} \\ \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \end{bmatrix} \end{aligned}$$

so the elements in the matrix can be easily identified (this is an example of the "system analysis" imaging task):

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We saw that $\underline{\mathbf{A}}^{-1}$ does not exist because the four rows (and the four columns) are not independent: the scalar product of $\underline{\mathbf{x}}$ with the fourth row conveys the same information as the first three.

Find the vector(s) in the null subspace by seeing which input vector $\underline{\mathbf{x}}$ satisfies the condition $\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{0}}$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_0 + x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{bmatrix} \Longrightarrow x_2 = -x_0, x_3 = -x_1, x_1 = -x_2$$

so one example of a null vector is:

$$\underline{\mathbf{x}}_n = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \Longrightarrow \text{the image is: } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Consider an arbitrary input image:

$$\underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow \underline{\mathbf{b}} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

Add this null vector to $\underline{\mathbf{x}}$:

$$\begin{aligned} \underline{\mathbf{x}} + \underline{\mathbf{x}}_n &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \\ &\Longrightarrow \quad \underline{\mathbf{b}} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

So:

$$\underline{\mathbf{A}}\underline{\mathbf{x}} = \underline{\mathbf{A}} \left(\underline{\mathbf{x}} + \underline{\mathbf{x}}_n \right)$$

This imaging system is not invertible, but we can change the system so that the inverse problem can be solved. We can add one or more other pieces of data: the sum of one or both diagonals. If we add only the diagonal $a + d$, the system matrix is:

$$\underline{\mathbf{A}}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \underline{\mathbf{b}} = \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \\ a+d \end{bmatrix}$$

which is not square, so the inverse $\underline{\mathbf{A}}_2^{-1}$ does not exist. However, the pseudoinverse matrix $\underline{\mathbf{A}}_2^{\dagger}$ DOES exist:

$$\underline{\mathbf{A}}_2^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\underline{\mathbf{A}}_2^T \underline{\mathbf{A}}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\left(\underline{\mathbf{A}}_2^T \underline{\mathbf{A}}_2 \right)^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

The pseudoinverse is:

$$\underline{\mathbf{A}}_2^\dagger \equiv \left(\underline{\mathbf{A}}_2^T \underline{\mathbf{A}}_2\right)^{-1} \underline{\mathbf{A}}_2^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{\underline{\mathbf{A}}_2^\dagger = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}}$$

Check by applying to some data:

$$\underline{\mathbf{A}}_2 \underline{\mathbf{x}} = \underline{\mathbf{b}} = \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \\ a+d \end{bmatrix}$$

$$\underline{\mathbf{A}}_2^\dagger \underline{\mathbf{b}} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a+c \\ b+d \\ a+b \\ c+d \\ a+d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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