SIMG-303-20033 Solution Set #6

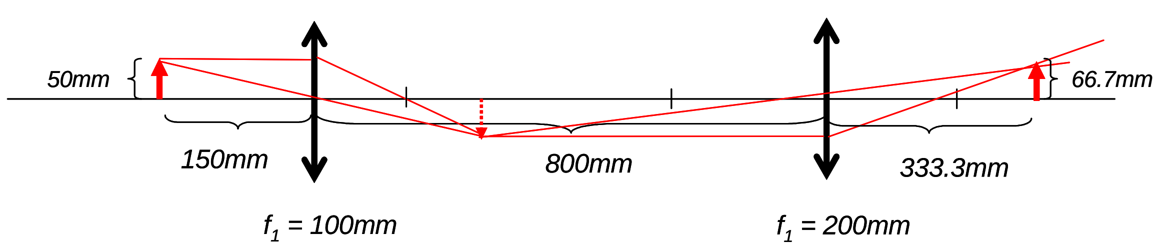
1. An optical system consists of thin lenses $L_1 (f_1 = 100\,\text{mm})$ and L_2 $(f_2 = 200\,\text{mm})$ separated by $d = 800\,\text{mm}$. Locate and describe the image of an object that is 50 mm high located 150 mm ”in front” of the first lens.

   At least three ways to do this. Easiest is probably by “brute force”, i.e., find the image created by the first lens and use it as the object for the second lens. The magnification of the system is the product of the magnifications of the two lenses.

$$\begin{aligned}
\frac{1}{s_1} + \frac{1}{s_1'} &= \frac{1}{f_1} \\
&\implies s_1^l = \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = \left(\frac{1}{100\,\text{mm}} - \frac{1}{150\,\text{mm}}\right)^{-1} \\
&= \left(\frac{3}{300\,\text{mm}} - \frac{2}{300\,\text{mm}}\right)^{-1} = \left(\frac{1}{300\,\text{mm}}\right)^{-1} \\
s_1' &= 300\,\text{mm} \\
(M_T)_1 &= -\frac{s_1'}{s_1} = -\frac{300\,\text{mm}}{150\,\text{mm}} = -2
\end{aligned}$$

$$\begin{aligned}
t &= s_1' + s_2 \\
800\,\text{mm} &= 300\,\text{mm} + s_2 \implies s_2 = 500\,\text{mm}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s_2} + \frac{1}{s_2'} &= \frac{1}{f_2} \\
&\implies s_2^l = \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{200\,\text{mm}} - \frac{1}{500\,\text{mm}}\right)^{-1} \\
&= \left(\frac{5}{1000\,\text{mm}} - \frac{2}{1000\,\text{mm}}\right)^{-1} = \left(\frac{3}{1000\,\text{mm}}\right)^{-1} \\
s_2' &= \frac{1000}{3}\,\text{mm} = 333\frac{1}{3}\,\text{mm} \\
(M_T)_2 &= -\frac{s_2'}{s_2} = -\frac{\left(\frac{1000}{3}\right)\,\text{mm}}{500\,\text{mm}} = -\frac{2}{3}
\end{aligned}$$



$$M_T = (M_T)_1 \cdot (M_T)_2 = -2 \cdot -\frac{2}{3} = +\frac{4}{3}$$

The image is located $\frac{1000}{3}$ mm *behind the second lens and the magnification is* $+\frac{4}{3}$, *so the image size is* $\frac{4}{3} \cdot 50\,\text{mm} = \frac{200}{3}\,\text{mm} = 66\frac{2}{3}\,\text{mm}$. *The image is upright and magnified.*

A second way to do this is is to locate the principal points of the lens and measure the object distance to the object-space principal point. Then use the thin-lens imaging equation to find the image distance, which is measured from the image-space principal point. The equivalent focal length of the system is:

$$f_{eff} = \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}\right)^{-1} = \frac{f_1 f_2}{(f_1 + f_2) - d} = \frac{100\,\text{mm} \cdot 200\,\text{mm}}{300\,\text{mm} - 800\,\text{mm}} = -40\,\text{mm}$$

The BFD is the distance from the rear vertex to the image-space focal point. The formula for the BFD was derived in the notes:

$$BFD = \overline{\mathbf{V'F'}} = \frac{f_2\,(f_1 - d)}{(f_1 + f_2) - d} = \frac{200\,\text{mm} \cdot (100\,\text{mm} - 800\,\text{mm})}{-500\,\text{mm}} = +280\,\text{mm}$$

The distance from the image-space principal point $\mathbf{H'}$ *to the image-space focal point* $\mathbf{F'}$ *is the effective focal length, so the distance from the image-space vertex to the image-space principal point can be found:*

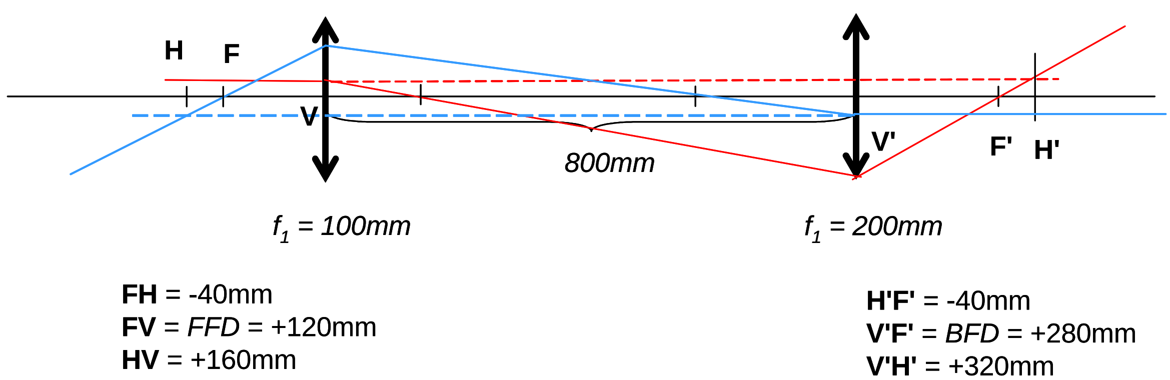
$$\overline{\mathbf{V'F'}} = \overline{\mathbf{V'F'}} - \overline{\mathbf{H'F'}} = 280\,\text{mm} - (-40\,\text{mm}) = 320\,\text{mm}$$

The object-space focal distance (FFD) may be found from the formula that was derived in the notes:

$$FFD = \overline{\mathbf{FV}} = \frac{f_1\,(f_2 - d)}{(f_1 + f_2) - d} = \frac{100\,\text{mm} \cdot (200\,\text{mm} - 800\,\text{mm})}{-500\,\text{mm}} = +120\,\text{mm}$$

The distance from the object-space vertex $\mathbf{V}$ *to the object-space principal point* $\mathbf{H}$ *can now be found:*

$$\begin{aligned} \overline{\mathbf{VH}} &= \overline{\mathbf{FH}} - \overline{\mathbf{FV}} = -40\,\text{mm} - 120\,\text{mm} = -160\,\text{mm} \\ &\Longrightarrow \overline{\mathbf{HV}} = +160\,\text{mm} \end{aligned}$$



The distance from the object to the object-space focal point is:

$$\begin{aligned} s &= \overline{\mathbf{OH}} = \overline{\mathbf{OV}} + \overline{\mathbf{VH}} \\ &= 150\,\text{mm} + (-160\,\text{mm}) = -10\,\text{mm} \end{aligned}$$

Since the object distance is negative, the object in the equivalent thin-lens equation is virtual. The image distance is now found directly:

$$\begin{aligned} \frac{1}{-10\,\text{mm}} + \frac{1}{s'} &= \frac{1}{-40\,\text{mm}} \\ s' &= \overline{\mathbf{H'O'}} = \left(\frac{1}{-40\,\text{mm}} - \frac{1}{-10\,\text{mm}}\right)^{-1} = +\frac{40}{3}\,\text{mm} = 13\frac{1}{3}\,\text{mm} \end{aligned}$$

The distance from the vertex to the image is obtained by substitution into:

$$\begin{aligned} s' &= \overline{\mathbf{H'O'}} = \overline{\mathbf{H'V'}} + \overline{\mathbf{V'O'}} \\ \overline{\mathbf{V'O'}} &= \overline{\mathbf{H'O'}} - \overline{\mathbf{H'V'}} \\ &= +\frac{40}{3}\,\text{mm} - (-320\,\text{mm}) = \frac{1000}{3}\,\text{mm} = 333\frac{1}{3}\,\text{mm} \end{aligned}$$

which is the same answer as that obtained above for s'_2*. The image magnification is the ratio of the distances:*

$$M_T = -\frac{s'}{s} = -\frac{+\frac{40}{3}\,\text{mm}}{-10\,\text{mm}} = +\frac{4}{3}$$

which again is the same answer as that obtained using the "brute-force" method.

2. A system consists of thin lenses L_1 ($f_1 = -60\,\mathrm{mm}$) and $L_2(f_2 =?)$ separated by $d = 120\,\mathrm{mm}$. Lens L_2 is made of glass with $n = 1.5$ and is plano-convex with radius $r = 60\,\mathrm{mm}$ for the curved side. Locate and describe the image of an object that is 5 mm high located 180 mm "in front" of the first lens.

   We need to first find the focal length of L_2. The focal length is determined from the curvatures of the surfaces and the index of refraction:

$$\begin{aligned}\frac{1}{f_2} &= (n_2 - n_{air})\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = 0.5 \cdot \left(\frac{1}{\infty} - \frac{1}{-60\,\mathrm{mm}}\right) = \frac{0.5}{60\,\mathrm{mm}} = \frac{1}{120\,\mathrm{mm}} \\ f_2 &= 120\,\mathrm{mm}\end{aligned}$$

$$\begin{aligned}\frac{1}{s_1} + \frac{1}{s_1'} &= \frac{1}{f_1} \\ s_1' &= \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = \left(\frac{1}{-60\,\mathrm{mm}} - \frac{1}{180\,\mathrm{mm}}\right)^{-1} = -45\,\mathrm{mm} \\ s_2 &= t - s_1' = 120\,\mathrm{mm} - (-45\,\mathrm{mm}) = 165\,\mathrm{mm} \\ s_2' &= \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{120\,\mathrm{mm}} - \frac{1}{165\,\mathrm{mm}}\right)^{-1} = +440\,\mathrm{mm}\end{aligned}$$

$$\begin{aligned}M_T &= (M_T)_1 \cdot (M_T)_2 \\ &= \left(-\frac{s_1'}{s_1}\right)\left(-\frac{s_2'}{s_2}\right) = \left(-\frac{-45\,\mathrm{mm}}{180\,\mathrm{mm}}\right)\left(-\frac{440\,\mathrm{mm}}{165\,\mathrm{mm}}\right) = -\frac{2}{3}\end{aligned}$$

   The image is located 440 mm *behind the second lens with a magnification of* $-\frac{2}{3}$, *so the image size is* $-\frac{10}{3}\,\mathrm{mm} = -3\frac{1}{3}\,\mathrm{mm}$

3. A thin lens is made of glass with index $n = 1.53$. In air, the lens has a focal length $f = 254\,\mathrm{mm}$. What is its focal length when it is totally immersed in water ($n = 1.33$)?

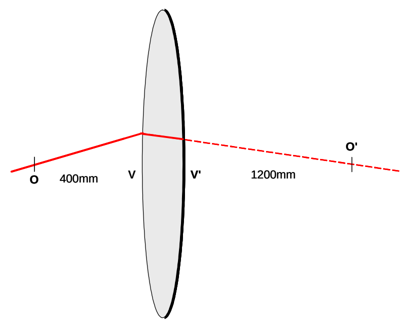
$$\begin{aligned}\textit{Lensmaker's equation} &: \quad \frac{1}{f} = (n_2 - n_1) \cdot \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ \textit{If in air} &: \quad \frac{1}{254\,\mathrm{mm}} = (1.53 - 1.0) \cdot \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ &= \quad 0.53 \cdot \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ &\Longrightarrow \left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{1}{0.53 \cdot 254\,\mathrm{mm}} = \frac{1}{134.62\,\mathrm{mm}} \\ \textit{If in water} &: \quad \frac{1}{f} = (1.53 - 1.33) \cdot \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ \frac{1}{f} &= 0.20 \cdot \frac{1}{0.53 \cdot 254\,\mathrm{mm}} \Longrightarrow \boxed{f = 673.08\,\mathrm{mm}}\end{aligned}$$

   The focal length of the lens in water is considerably longer than its focal length in air because the "relative refractivity" of the glass is much reduced in water. To see that this result makes sense, consider what the focal length of the lens would be if immersed in glass. The "relative refractivity" of the lens vanishes, so the focal length becomes infinite.

4. The surfaces of a thin equiconvex lens have equal radius of curvature: $|R_1| = 150\,\text{mm}$. The second surface is aluminized so that it is a mirror. Find the location of the image of an object located 400 mm to the left of the first surface.

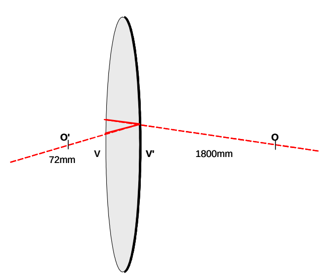
We can do this in three steps: refraction at the first surface, reflection at the second surface, and refraction back at the first surface. Assume that the refractive index of the glass is $n = 1.5$:

$$\begin{aligned}
\frac{n_1}{s_1} + \frac{n_2}{s_1'} &= \frac{n_2 - n_1}{R_1} = \frac{1.5 - 1}{R_1} = \frac{0.5}{150\,\text{mm}} = \frac{1}{300\,\text{mm}} \\
\frac{s_1'}{n_2} &= \left(\frac{1}{300\,\text{mm}} - \frac{1}{400\,\text{mm}}\right)^{-1} = \left(\frac{4}{1200\,\text{mm}} - \frac{3}{1200\,\text{mm}}\right)^{-1} \\
&= \left(\frac{1}{1200\,\text{mm}}\right)^{-1} \Longrightarrow s_1' = n_2 \cdot 1200\,\text{mm} = 1800\,\text{mm}
\end{aligned}$$



Because the lens is thin, the distance from the first to the second surface is 0 mm. *For a mirrored surface, the image-space index* $n_2 = -n_1$:

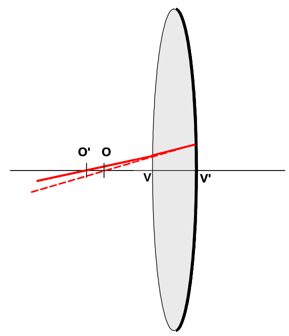
$$\begin{aligned}
\frac{n_2}{s_2} + \frac{n_3}{s_2'} &= \frac{1.5}{-1800\,\text{mm}} + \frac{-1.5}{s_2'} = \frac{(-1.5 - 1.5)}{R_2} = \frac{-3}{-150\,\text{mm}} = +\frac{1}{50\,\text{mm}} \\
\frac{s_2'}{1.5} &= \left(\frac{1}{50\,\text{mm}} - \frac{1}{-1200\,\text{mm}}\right)^{-1} = \left(\frac{24}{1200} + \frac{1}{1200}\right)^{-1} = \frac{1200\,\text{mm}}{25} = +48\,\text{mm} \\
s_2' &= 1.5 \cdot +48\,\text{mm} = 72\,\text{mm}
\end{aligned}$$



The image is 48 mm *"behind" the second surface, so* $s_3 = -48\,\text{mm}$:

$$\begin{aligned}\frac{1.5}{s_3} + \frac{1}{s_3'} &= \frac{(1-1.5)}{-150\,\text{mm}} \\ \frac{1}{s_3'} &= \left(\frac{1}{300\,\text{mm}} - \frac{1.5}{72\,\text{mm}}\right)^{-1} = -\frac{400}{7}\,\text{mm} \simeq -57.14\,\text{mm}\end{aligned}$$

This is "behind" the second surface, but since we are going in the negative direction, the distance is "in front of" the lens. In other words, the distance is "positive", so that $s_3' = +\frac{400}{7}\,\text{mm}$ *if measured in the same space as the object.*



The image is real and the magnification is

$$M_T = -\frac{s_3'}{s_1} = -\frac{\frac{400}{7}\,\text{mm}}{400\,\text{mm}} = -\frac{1}{7}$$

5. Two thin lenses of focal lengths $f_1 = -50\,\text{mm}$ and $f_2 = +100\,\text{mm}$ are separated by a distance $d = 50\,\text{mm}$. Find the focal length of the system of thin lenses and locate the principal points.

The focal length of the system may be found from:

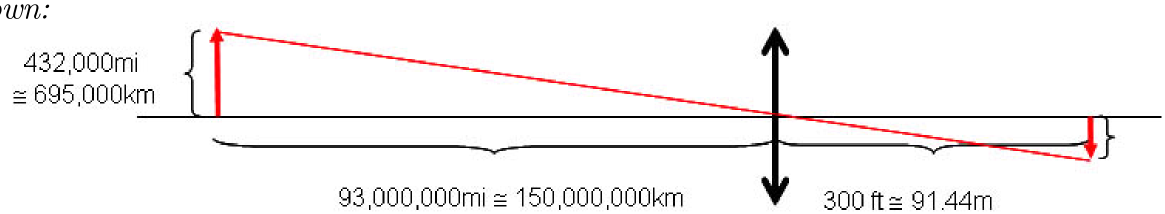
$$
\begin{aligned}
f_{eff} &= \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}\right)^{-1} \\
&= \left(\frac{1}{-50\,\text{mm}} + \frac{1}{100\,\text{mm}} - \frac{50\,\text{mm}}{(-50\,\text{mm} \cdot 100\,\text{mm})}\right)^{-1} \\
&= \left(\frac{2}{-100\,\text{mm}} + \frac{1}{100\,\text{mm}} + \frac{1}{100\,\text{mm}}\right)^{-1} = 0^{-1}
\end{aligned}
$$

The system is AFOCAL; it is a telescope.

The focal length is ∞ and all cardinal points (focal and principal points) are located at ∞.

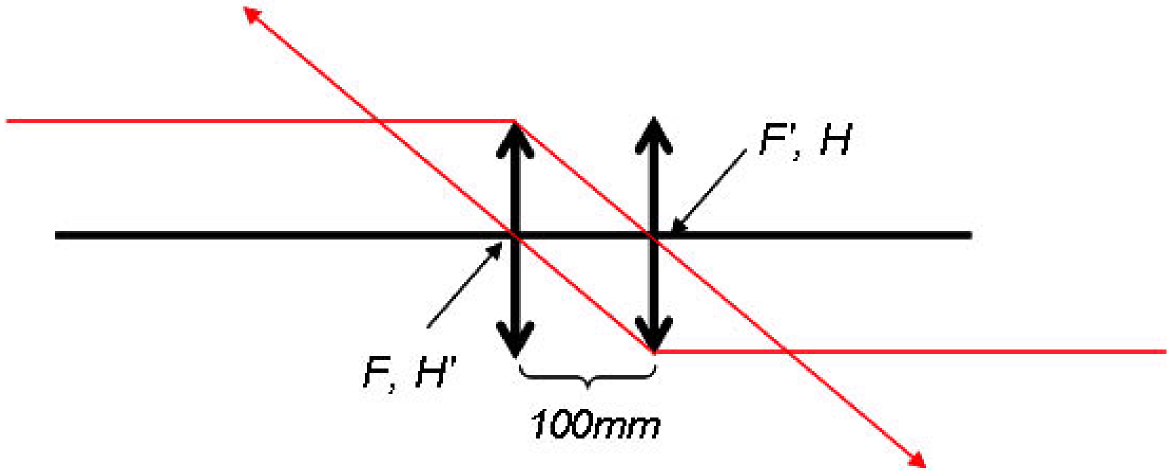
6. The solar telescope at the Kitt Peak National Observatory has a primary mirror with a diameter of 60 in. The image formed by this mirror is 300 ft away. If the diameter of the sun is 864,000 mi and its distance is 93,000,000 mi. Find the diameter of the image of the sun in mm.

The image size is proportional to the ratio of the object distance to the focal length, as shown:



$$
\begin{aligned}
\frac{864,000\,\text{mi}}{93,000,000\,\text{mi}} &= \frac{x}{300\,\text{ft}} \\
x &= \frac{864,000\,\text{mi}}{93,000,000\,\text{mi}} \cdot 300\,\text{ft} \\
&\simeq 2.787\,\text{ft} \simeq 850\,\text{mm}
\end{aligned}
$$

7. Consider an optical system composed of two thin lenses, each with focal length of 100 mm. The first lens is located at the front focal point of the second. Find the focal length, focal points, and the principal points of the system. If the focal length of the first lens is changed, determine the effect on these parameters of the system.

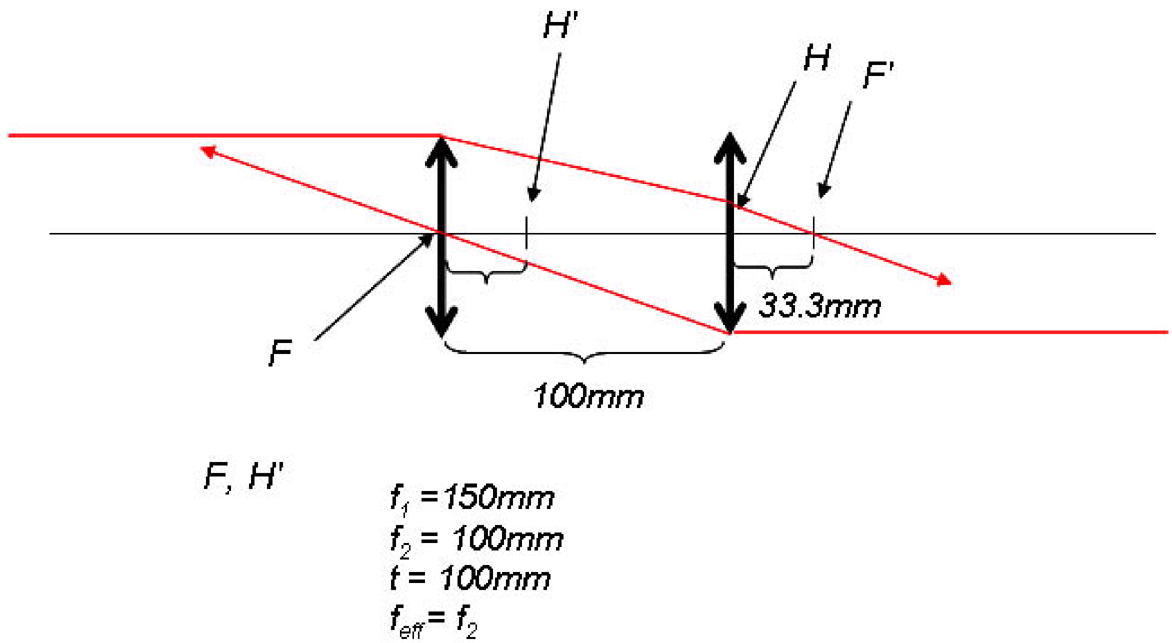


$f_1 = f_2 = 100mm$
$t = 100mm$

The focal length of the system is:

$$\begin{aligned}
\frac{1}{f_{eff}} &= \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \\
\frac{1}{f_{eff}} &= \frac{1}{100\,\text{mm}} + \frac{1}{100\,\text{mm}} - \frac{100\,\text{mm}}{100\,\text{mm} \cdot 100\,\text{mm}} \\
&= \frac{1}{100\,\text{mm}} \\
&\Longrightarrow f_{eff} = 100\,\text{mm}
\end{aligned}$$

The focal length of the system is the same as the focal length of each lens. The image-space focal point is located at the second lens. The object-space focal point is located at the first lens. Since the lenses are located f_{eff} apart, then the principal points coincide with the lenses, as shown in the sketch.



If the focal length of the first lens is changed, say $f_1 = 150\,\text{mm}$, *then the focal length of the system is:*

$$\begin{aligned}
\frac{1}{f_{eff}} &= \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \\
\frac{1}{f_{eff}} &= \frac{1}{150\,\text{mm}} + \frac{1}{100\,\text{mm}} - \frac{100\,\text{mm}}{150\,\text{mm} \cdot 100\,\text{mm}} \\
&= \frac{1}{100\,\text{mm}} \\
&\Longrightarrow f_{eff} = 100\,\text{mm} = f_2
\end{aligned}$$

Thus the focal length of the system is equal to that of the second lens, regardless of the focal length of the first lens.

(a) (OPTIONAL BONUS) Describe any applications for this optical system.

Consider the image formed by the second lens in the second system with $f_2 = 100\,\text{mm}$ *of a nearby object, say at* $s_1 = 500\,\text{mm}$. *The image formed by the single lens is located at the distance:*

$$s' = \left(\frac{1}{f_2} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{100\,\text{mm}} - \frac{1}{500\,\text{mm}}\right)^{-1} = \left(\frac{4}{500\,\text{mm}}\right)^{-1} = 125\,\text{mm}$$

The magnification is

$$M_T = -\frac{s'}{s} = -\frac{125\,\text{mm}}{500\,\text{mm}} = -\frac{1}{4}$$

Now add the first lens. The focal length of the system is unchanged at $f_{eff} = f_2 = 100\,\text{mm}$. *Now find the image of an object located* $500\,\text{mm}$ *in front of the SECOND lens (i.e., in the same position as before) and find its magnification. Do it brute* *force:*

$$\begin{aligned}
s_1 &= 500\,\text{mm} - t = 500\,\text{mm} - 100\,\text{mm} = 400\,\text{mm} \\
\frac{1}{s_1} + \frac{1}{s_1'} &= \frac{1}{f_1} \Longrightarrow s_1' = \left(\frac{1}{150\,\text{mm}} - \frac{1}{400\,\text{mm}}\right)^{-1} = 240\,\text{mm} \\
(M_T)_1 &= -\frac{s_1'}{s_1} = -\frac{240\,\text{mm}}{400\,\text{mm}} = -0.6
\end{aligned}$$

The distance to the second lens is:

$$\begin{aligned}
s_2 &= t - s_1' \\
&= 100\,\text{mm} - 240\,\text{mm} = -140\,\text{mm} \\
s_2' &= \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{100\,\text{mm}} - \frac{1}{-140\,\text{mm}}\right)^{-1} = \frac{175}{3}\,\text{mm} = 58\frac{1}{3}\,\text{mm} \\
(M_T)_2 &= -\frac{s_2'}{s_2} = -\frac{\frac{175}{3}\,\text{mm}}{-140\,\text{mm}} = +\frac{5}{12}
\end{aligned}$$

The transverse magnification of the image formed by the system is the product:

$$M_T = (M_T)_1 \cdot (M_T)_2 = -\frac{3}{5} \cdot +\frac{5}{12} = -\frac{1}{4}$$

The image of the object formed by the first lens alone is located $125\,\text{mm}$ *behind the lens with a magnification of* $-\frac{1}{4}$*, whereas the image formed by both lenses is located* $58\frac{1}{3}\,\text{mm}$ *behind the second lens with the same magnification. The addition of the first lens located at the front focal point of the second lens moved the image without changing its magnification. This is the action of eyeglasses – the optometrist locates a corrective lens at the front focal point of the eyelens to move the image onto the retina without changing the system magnification. In the case just considered, the "corrective" lens* L_1 *has positive power and moved the image "forward", as required for a farsighted person. Had the power of* L_1 *been negative, the image would have moved "backward" onto the retina of a nearsighted person.*