

# SIMG-303-20033      Solution Set #3

1. A pulse of light from a sodium lamp with  $\lambda = 589 \text{ nm}$  passes through a tank of glycerine that is 20 m long in time  $t_1$ . When the same tank is filled with carbon disulfide, the time required for the pulse of light to traverse the tank is  $t_2$ . The refractive indices of the two media are  $n_1 = 1.47$  and  $n_2 = 1.63$ .

- (a) Determine the propagation times and the time difference  $t_1 - t_2$ .

*We know that:*

$$s = 20 \text{ m} = v \cdot t = \frac{c}{n}t \implies t = \frac{n}{c}s$$

$$t_1 = n_1 \cdot \frac{20 \text{ m}}{3 \cdot 10^8 \frac{\text{m}}{\text{s}}} = 1.47 \cdot \frac{20 \text{ m}}{3 \cdot 10^8 \frac{\text{m}}{\text{s}}} = 98 \times 10^{-9} \text{ s} \simeq 100 \text{ ns}$$

$$t_2 = 1.637 \cdot \frac{20 \text{ m}}{3 \cdot 10^8 \frac{\text{m}}{\text{s}}} = 109.13 \times 10^{-9} \text{ s} \simeq 110 \text{ ns}$$

$$\Delta t = t_1 - t_2 = 98 \times 10^{-9} \text{ s} - 109.13 \times 10^{-9} \text{ s} = -11.13 \times 10^{-9} \text{ s} \simeq -11 \text{ ns}$$

*FYI, Light travels just about 1 foot in 1 nanosecond, so it travels 100 and 110 feet in these times. The negative sign on  $\Delta t$  just means that we referenced the measurement to the shorter time*

- (b) If one pulse of light with the same wavelength is divided in two and passed through two tanks, one with glycerine and one with carbon disulfide, describe the pulses at the output ends of the tanks.

*The two pulses are separated in time by  $11.13 \times 10^{-9} \text{ s}$ , so they are separated in space by*

$$s = ct = 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \cdot 11.13 \times 10^{-9} \text{ s} = 3.34 \text{ m} \simeq 11 \text{ ft}$$

*The pulse through the tank of glycerine leads the pulse through the tank of carbon disulfide by this distance*

2. Determine  $k_0$  and  $\omega_0$  for light with  $\lambda = 600 \text{ nm}$  in two situations:

- (a) in vacuum

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{600 \text{ nm}} = 1.0472 \times 10^7 \frac{\text{radians}}{\text{m}} \text{ (this is the "wavenumber")}$$

$$\frac{\omega}{k} = c \implies \omega = ck = 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \cdot 1.0472 \times 10^7 \frac{\text{radians}}{\text{m}} = 3.1416 \times 10^{15} \frac{\text{radians}}{\text{s}}$$

- (b) in glass with  $n = 1.5$ .

$$\lambda' = \frac{\lambda}{n} = \frac{600 \text{ nm}}{1.5} = 400.0 \text{ nm}$$

$$k' = \frac{2\pi}{\lambda'} = \frac{2\pi}{400 \text{ nm}} = 6.9813 \times 10^6 \frac{\text{radians}}{\text{m}}$$

$$\frac{\omega}{k} = v = \frac{c}{n} \implies \omega = vk' = 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \cdot 1.0472 \times 10^7 \frac{\text{radians}}{\text{m}} = 3.1416 \times 10^{15} \frac{\text{radians}}{\text{s}}$$

3. The phase velocity of waves in some medium is proportional to  $\omega^{+\frac{1}{4}}$ . Find an expression for the modulation velocity and determine whether the waves exhibit normal or anomalous dispersion.

$$v_\phi = \frac{c}{n} = \frac{\omega}{k} = \alpha\omega^{+\frac{1}{4}} \text{ where } \alpha \text{ is some constant}$$

$$\implies k = \frac{\omega^{+\frac{3}{4}}}{\alpha} \implies \omega^{+\frac{3}{4}} = \alpha k \implies \boxed{\omega = (\alpha k)^{+\frac{4}{3}}}$$

*This is the dispersion relation  $\omega [k]$*

$$v_{\text{mod}} = \frac{d\omega}{dk} = \alpha^{+\frac{4}{3}} \cdot +\frac{4}{3}k^{+\frac{1}{3}} = +\frac{4}{3}\alpha^{+\frac{4}{3}} \cdot k^{+\frac{1}{3}} = +\frac{4}{3}\alpha^{+\frac{4}{3}} \cdot \left(\frac{\omega^{+\frac{3}{4}}}{\alpha}\right)^{+\frac{1}{3}}$$

$$= +\frac{4}{3}\alpha^{+\frac{4}{3}} \cdot \alpha^{-\frac{1}{3}}\omega^{+\frac{1}{4}}$$

$$v_{\text{mod}} = \frac{4}{3}\alpha\omega^{+\frac{1}{4}} > \alpha\omega^{+\frac{1}{4}} = v_\phi \implies \text{anomalous dispersion}$$

4. The variation in refractive index with wavelength for a transparent material (such as glass) may be approximately expressed in terms of an empirical equation (*Cauchy's equation*)

$$n[\lambda_0] = A + \frac{B}{\lambda_0^2}$$

where  $A$  and  $B$  are constants derived from experimental measurements and  $\lambda_0$  is the vacuum wavelength of the incident light.

- (a) What are the dimensions (units) of  $A$  and  $B$ ?

*Since  $n$  is dimensionless (i.e., a "pure" number), then  $A$  is dimensionless*

*Therefore  $\frac{B}{\lambda_0^2}$  is dimensionless, and  $B$  must have units of  $[\text{Length}]^{-2}$ , e.g.,  $\text{m}^{-2}$*

- (b) Find an expression for the group velocity for  $\lambda_0 = 500 \text{ nm}$  in glass with:

$$n[\lambda_0] = 1.5 + \frac{3 \times 10^{+4}}{\lambda_0^2}$$

where  $\lambda_0$  is measured in nm.

$$n[500 \text{ nm}] = 1.5 + \frac{3 \times 10^{+4}}{(500)^2} = 1.62$$

*Check to see what the index is at a shorter wavelength, say  $\lambda_0 = 490 \text{ nm}$ :*

$$n[490 \text{ nm}] = 1.5 + \frac{3 \times 10^{+4}}{(490)^2} = 1.6249$$

*which confirms that shorter wavelengths have larger indices of refraction and thus travel more slowly. Thus the dispersion is normal. But this doesn't tell us the*

group velocity – we need to convert this relation for  $n$  into the form  $\omega [k]$ , so that we can find:

$$\begin{aligned}
 v_\phi &= \frac{\omega}{k} = \frac{c}{n} \implies \omega = \frac{ck}{n} \implies \frac{d\omega}{dk} = \frac{d}{dk} \left( \frac{ck}{n} \right) = \frac{c}{n} + ck \cdot \frac{d}{dk} \left( \frac{1}{n} \right) \\
 &= \frac{c}{n} + ck \cdot \left( -\frac{1}{n^2} \frac{dn}{dk} \right) \\
 &= \frac{c}{n} - \frac{c}{n} \cdot \left( \frac{k}{n} \frac{dn}{dk} \right) \\
 &= \frac{c}{n} \left( 1 - \frac{k}{n} \frac{dn}{dk} \right) \\
 v_{\text{mod}} &= v_\phi \left( 1 - \frac{k}{n} \frac{dn}{dk} \right)
 \end{aligned}$$

So  $v_\phi > v_{\text{mod}}$  (normal dispersion) exists if  $\frac{dn}{dk} > 0$ , which means that  $\frac{dn}{d\lambda} < 0$ . We can convert this to an expression in terms of  $\lambda_0$  by appropriate substitution:

$$\begin{aligned}
 k &= \frac{2\pi}{\lambda_0} \\
 \frac{dn}{dk} &= \frac{dn}{d\lambda_0} \cdot \frac{d\lambda_0}{dk} = \frac{d\lambda_0}{dk} \cdot \frac{dn}{d\lambda_0}
 \end{aligned}$$

$$\begin{aligned}
 v_{\text{mod}} &= v_\phi \left( 1 - \frac{2\pi}{n\lambda_0} \left( \frac{d\lambda_0}{dk} \right) \left( \frac{dn}{d\lambda_0} \right) \right) \\
 &= v_\phi \left( 1 - \frac{2\pi}{n\lambda_0} \left( -\frac{\lambda_0^2}{2\pi} \right) \left( \frac{dn}{d\lambda_0} \right) \right) \\
 &= v_\phi \left( 1 + \frac{\lambda_0}{n} \left( \frac{dn}{d\lambda_0} \right) \right) \\
 &= v_\phi \left( 1 + \frac{\lambda_0}{n} \frac{d}{d\lambda_0} \left( 1.5 + \frac{3 \times 10^{+4}}{\lambda_0^2} \right) \right) \\
 &= v_\phi \left( 1 + \frac{\lambda_0}{n} \frac{d}{d\lambda_0} \cdot (-2\lambda_0^{-3} \cdot 3 \times 10^{+4}) \right) \\
 &= v_\phi \left( 1 + \frac{\lambda_0}{n} \frac{d}{d\lambda_0} \cdot (-2\lambda_0^{-3} \cdot 3 \times 10^{+4}) \right) \\
 &= v_\phi \left( 1 - \frac{6 \times 10^{+4}}{n\lambda_0^2} \right)
 \end{aligned}$$

At  $\lambda_0 = 500 \text{ nm}$

$$v_{\text{mod}} = v_\phi \left( 1 - \frac{6 \times 10^{+4}}{1.62 \cdot 500^2} \right) \simeq 0.85v_\phi$$

5. The refractive index of a certain hypothetical material is found to vary as the reciprocal of the vacuum wavelength. Find an expression for the group velocity in terms of the phase velocity.

$$\begin{aligned}
 n[\lambda_0] &= \frac{\alpha}{\lambda_0} = \frac{\alpha}{2\pi} \cdot \frac{2\pi}{\lambda_0} = \frac{\alpha}{2\pi} \cdot k \\
 v_\phi &= \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\left(\frac{\alpha}{2\pi}k\right)} = \frac{2\pi c}{\alpha k} \\
 \frac{\omega}{k} &= \frac{c}{n} \implies \omega = \frac{ck}{n} \\
 \implies \frac{d\omega}{dk} &= \frac{d}{dk} \left( \frac{ck}{n} \right) = \frac{c}{n} + ck \cdot \frac{d}{dk} \left( \frac{1}{n} \right) \\
 \frac{d\omega}{dk} &= \frac{c}{n} + ck \cdot \left( -\frac{1}{n^2} \frac{dn}{dk} \right) \\
 &= \frac{c}{n} - \frac{c}{n} \cdot \left( \frac{k}{n} \frac{dn}{dk} \right) \\
 &= \frac{c}{n} \left( 1 - \frac{k}{n} \frac{dn}{dk} \right) = v_\phi \left( 1 - \frac{k}{n} \frac{dn}{dk} \right) \\
 \frac{d\omega}{dk} &= v_\phi \left( 1 - \frac{k}{\left(\frac{\alpha}{2\pi}k\right)} \frac{d}{dk} \left( \frac{\alpha}{2\pi}k \right) \right) \\
 &= v_\phi \left( 1 - \frac{1}{\left(\frac{\alpha}{2\pi}\right)} \frac{\alpha}{2\pi} \right) = v_\phi (1 - 1) = 0
 \end{aligned}$$