

1. $N > 1$ simple temporal harmonic oscillatory motions with the same amplitude and temporal frequency are superimposed (summed), i.e., the output $g[t]$ may be written as:

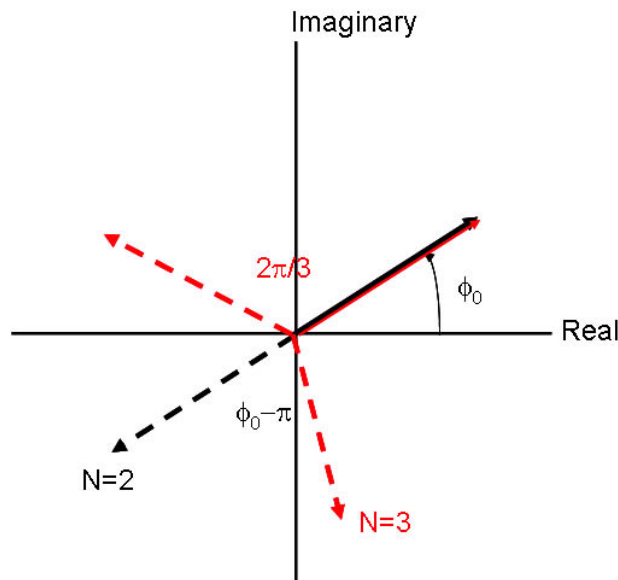
$$g[t] = \sum_{n=1}^N A_0 \cos[2\pi\nu_0 t + \phi_n]$$

The phase difference between successive pairs of oscillations is:

$$\Delta\phi_n = \phi_n - \phi_{n-1}, \text{ where } n = 2, 3, \dots, N$$

If the phase difference between each successive pair is invariant, find one value of this phase difference as a function of N for which the amplitude of the sum is zero. Again, it may be useful to draw a picture to help you solve the problem, and it also may be useful to consider the result for small integer values of N .

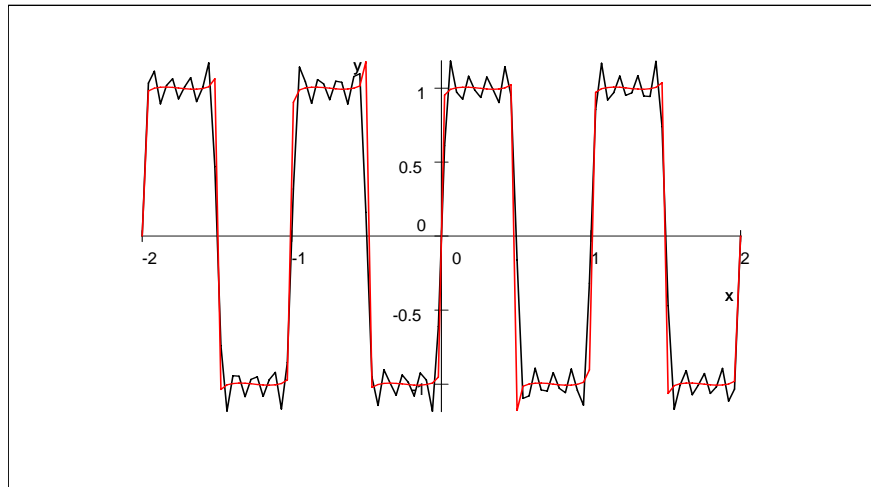
Easy to see on the Argand diagram. The initial phase for the first oscillator is ϕ_1 . If $N = 2$, then the second oscillator must have the same amplitude but be “out of phase” relative to the first, i.e., $\phi_2 = \phi_1 \pm \pi$, as shown on the sketch. If $N = 3$, then the sum of the second and third oscillations must create a term with the same amplitude and “out of phase”,. In general, the initial phase of the second oscillation could have the angle $\phi_1 + \frac{2\pi}{N}$, where N is the number of oscillators to be added.



2. Sketch the following functions and their sum over a domain at least as large as $-2 \leq x \leq +2$. In each case, estimate the function that would be obtained if the upper limit of the sum is $+\infty$. You may use computer software if desired (and it likely will be easier to do so), but this is not required.

(a)

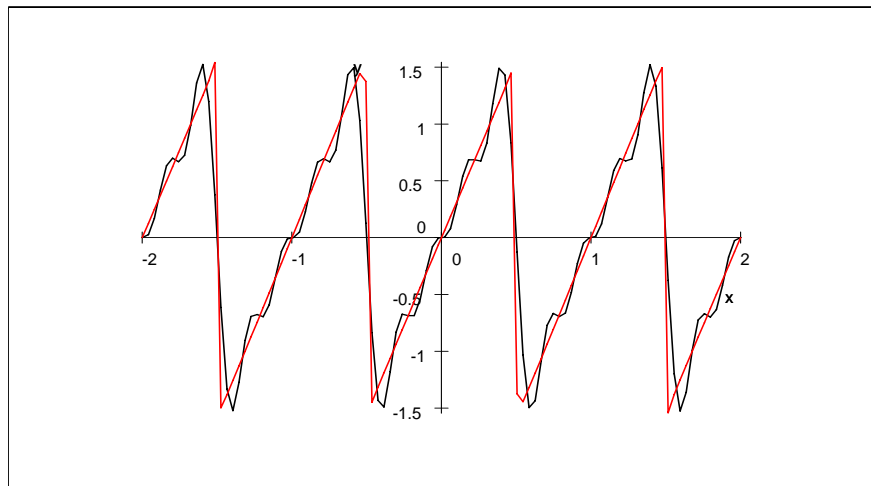
$$f_1[x] = \sum_{n=0}^3 \frac{4}{(2n+1)\pi} \sin [2\pi (2n+1)x]$$



Black: $f_1[x] = \sum_{n=0}^3 \frac{4}{(2n+1)\pi} \sin [2\pi (2n+1)x]$, Red
 Dashes: $f_1[x] = \sum_{n=0}^{50} \frac{4}{(2n+1)\pi} \sin [2\pi (2n+1)x]$

(b)

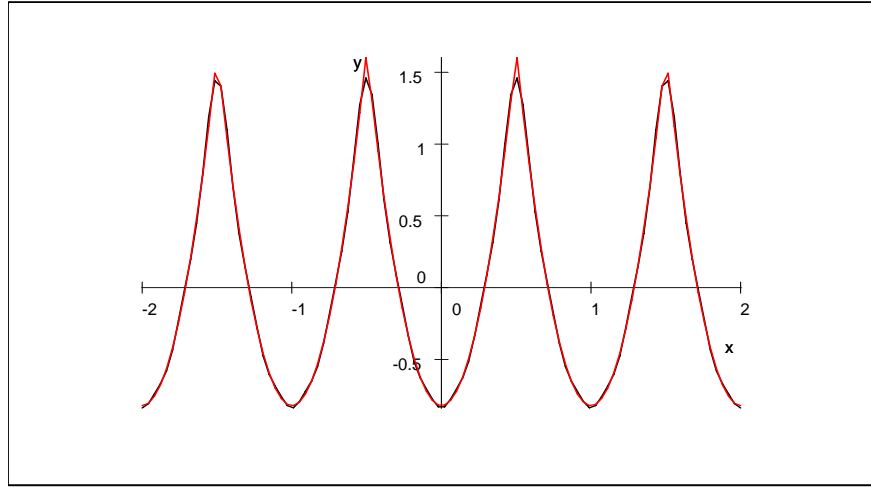
$$f_2[x] = \sum_{n=1}^4 (-1)^{n+1} \left(\frac{1}{n} \sin [2\pi nx] \right)$$



Black: $\sum_{n=1}^4 (-1)^{n+1} \left(\frac{1}{n} \sin [2\pi nx] \right)$, Red Dashes: $\sum_{n=1}^{50} (-1)^{n+1} \left(\frac{1}{n} \sin [2\pi nx] \right)$, Sawtooth Wave

(c)

$$f_3[x] = \sum_{n=1}^5 (-1)^n \left(\frac{1}{n^2} \cos [2\pi n x] \right)$$



Black: $\sum_{n=1}^5 (-1)^n \left(\frac{1}{n^2} \cos [2\pi n x] \right)$, Red Dashes: $\sum_{n=1}^{50} (-1)^n \left(\frac{1}{n^2} \cos [2\pi n x] \right)$

3. Consider the sum of two traveling waves with different wavelengths and temporal frequencies:

$$f_n [z, t] = \cos \left[2\pi \left(\frac{z}{\lambda_n} - \nu_n t \right) + \phi_n \right] = \cos [k_n z - \omega_n t + \phi_n]$$

where $n = 1, 2$. The parameters are $\lambda_1 = 400 \text{ mm}$, $\nu_1 = 800 \text{ Hz}$, $\phi_1 = \frac{\pi}{2}$ radians, $\lambda_2 = 600 \text{ mm}$, $\nu_2 = 1200 \text{ Hz}$, $\phi_2 = -\frac{\pi}{4}$ radians

4. Derive the expression for the sum of these two waves.

$$\begin{aligned} v_1 &= \frac{\omega_1}{k_1} = \frac{2\pi (800 \text{ Hz})}{\frac{2\pi}{400 \text{ mm}}} = 320,000 \frac{\text{mm}}{\text{s}} = 320 \frac{\text{m}}{\text{s}} \\ v_2 &= \frac{\omega_2}{k_2} = \frac{2\pi (1200 \text{ Hz})}{\frac{2\pi}{600 \text{ mm}}} = 720,000 \frac{\text{mm}}{\text{s}} = 720 \frac{\text{m}}{\text{s}} \\ \lambda_1 &= 400 \text{ mm} < \lambda_2 = 600 \text{ mm} \end{aligned}$$

So the longer wavelength travels faster.

From class :

$$\begin{aligned} f_1 [z, t] + f_2 [z, t] &= \cos [k_1 z - \omega_1 t + \phi_1] + \cos [k_2 z - \omega_2 t + \phi_2] \\ &= 2 \cos \left[\left(\frac{k_1 + k_2}{2} \right) z + \left(\frac{\omega_1 + \omega_2}{2} \right) t + \left(\frac{\phi_1 + \phi_2}{2} \right) \right] \\ &\quad \cdot \cos \left[\left(\frac{k_1 - k_2}{2} \right) z + \left(\frac{\omega_1 - \omega_2}{2} \right) t + \left(\frac{\phi_1 - \phi_2}{2} \right) \right] \\ &\equiv 2 \cos [k_{avg} z + \omega_{avg} t + \phi_{avg}] \cos [k_{mod} z + \omega_{mod} t + \phi_{mod}] \end{aligned}$$

$$f_1 [z, t] + f_2 [z, t] = \cos \left[2\pi \frac{z}{400 \text{ mm}} - 2\pi (800 \text{ Hz}) t + \frac{\pi}{2} \right] + \cos \left[2\pi \frac{z}{600 \text{ mm}} - 2\pi (1200 \text{ Hz}) t - \frac{\pi}{4} \right]$$

$$k_{avg} = \frac{1}{2} \cdot \left(\frac{2\pi}{400 \text{ mm}} + \frac{2\pi}{600 \text{ mm}} \right) = \frac{2\pi}{480 \text{ mm}} \implies \lambda_{avg} = 480 \text{ mm}$$

$$k_{mod} = \frac{1}{2} \cdot \left(\frac{2\pi}{400 \text{ mm}} - \frac{2\pi}{600 \text{ mm}} \right) = \frac{2\pi}{2400 \text{ mm}} \implies \lambda_{mod} = 2400 \text{ mm}$$

$$\omega_{avg} = \frac{1}{2} (2\pi \cdot 800 \text{ Hz} + 2\pi \cdot 1200 \text{ Hz}) = 2\pi \cdot 1000 \text{ Hz} \implies \nu_{avg} = 1000 \text{ Hz}$$

$$\omega_{mod} = \frac{1}{2} (2\pi \cdot 800 \text{ Hz} - 2\pi \cdot 1200 \text{ Hz}) = -2\pi \cdot 200 \text{ Hz} \implies \nu_{avg} = -200 \text{ Hz}$$

$$\phi_{avg} = \frac{1}{2} \left(+\frac{\pi}{2} - \frac{\pi}{4} \right) = +\frac{\pi}{8}$$

$$\phi_{mod} = \frac{1}{2} \left(+\frac{\pi}{2} - \left(-\frac{\pi}{4} \right) \right) = +\frac{3\pi}{8}$$

- (a) Derive the “phase velocity” and “group velocity” (also called the “modulation velocity”) of these waves.

$$\begin{aligned} v_\phi &= \frac{\omega_{avg}}{k_{avg}} = \frac{2\pi \cdot 1000 \text{ Hz}}{\left(\frac{2\pi}{480 \text{ mm}} \right)} = 480,000 \frac{\text{mm}}{\text{s}} = 480 \frac{\text{m}}{\text{s}} \\ v_{mod} &= \frac{\omega_{mod}}{k_{mod}} = \frac{-2\pi \cdot 200 \text{ Hz}}{\left(\frac{2\pi}{2400 \text{ mm}} \right)} = -480,000 \frac{\text{mm}}{\text{s}} = -480 \frac{\text{m}}{\text{s}} \end{aligned}$$

- (b) Determine if these waves exhibit normal or anomalous dispersion or are not dispersive, and explain why.

$v_1 < v_2 \implies v_\phi > v_{\text{mod}}$, and the waves exhibit normal dispersion.

- (c) Change one of the parameters in the list twice to create waves that exhibit the other two kinds of dispersion, i.e., if the original parameters result in normal dispersion, change one of the parameters so that the waves exhibit anomalous dispersion and change it again to make the waves nondispersive.

To make nondispersive waves, we need to have $v_1 = v_2$, I'll change k_2 to make it k'_2

$$\begin{aligned}\frac{\omega_1}{k_1} &= \frac{\omega_2}{k'_2} \\ \frac{\omega_1}{k_1} &= \frac{2\pi(800\text{ Hz})}{\frac{2\pi}{400\text{ mm}}} = 320,000 \frac{\text{mm}}{\text{s}} = 320 \frac{\text{m}}{\text{s}} \\ \frac{\omega_2}{k'_2} &= \frac{2\pi(1200\text{ Hz})}{\frac{2\pi}{\lambda'_2}} = 1200 \cdot \lambda'_2 = 320 \frac{\text{m}}{\text{s}} \\ \lambda'_2 &= \frac{320 \frac{\text{m}}{\text{s}}}{1200 \frac{1}{\text{s}}} = \frac{4}{15} \text{ m} \simeq 267 \text{ mm vs, } \lambda_2 = 600 \text{ mm}\end{aligned}$$

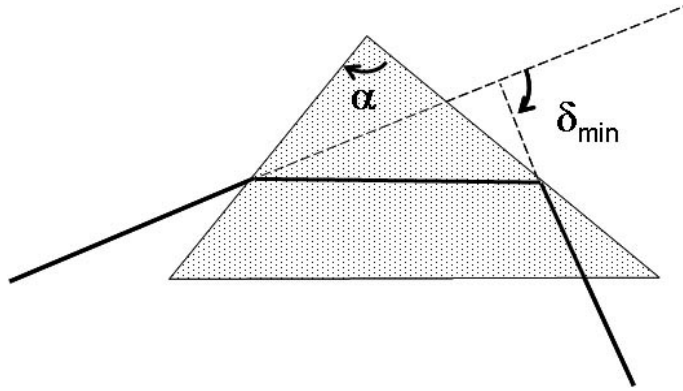
To make anomalous dispersion, we need to have $v_1 = \frac{\omega_1}{k_1} > v_2 = \frac{\omega_2}{k_2}$

$$\begin{aligned}\frac{\omega_1}{k_1} > \frac{\omega_2}{k_2} &\implies k'_2 > \frac{\omega_2}{\omega_1} k_1 \implies \frac{1}{\lambda'_2} > \frac{\omega_2}{\omega_1} \frac{1}{\lambda_1} \implies \frac{\omega_1}{\omega_2} \lambda_1 > \lambda'_2 \\ \frac{\omega_1}{\omega_2} \cdot \lambda_1 > \lambda'_2 &= \frac{2\pi(800\text{ Hz})}{2\pi(1200\text{ Hz})} \cdot 400\text{ mm} = \frac{800}{3} \text{ mm} \simeq 267 \text{ mm} \\ \text{so } \lambda'_2 > 267 \text{ mm}\end{aligned}$$

5. OPTIONAL BONUS PROBLEM: In the lab on dispersion, we presented the equation for the angle of minimum deviation δ_{\min} for a prism with refractive index n and apex angle α :

$$n = \frac{\sin \left[\frac{\delta_{\min} + \alpha}{2} \right]}{\sin \left[\frac{\alpha}{2} \right]}$$

If the prism is made of glass with $n = 1.5$, determine the smallest angle α for which no light is transmitted through the prism. You will need the formula for the sine of the sum of two angles.



$$\begin{aligned} 1.5 \cdot \sin \left[\frac{\alpha}{2} \right] &= \sin \left[\frac{\delta_{\min} + \alpha}{2} \right] \\ \sin \left[\frac{\delta_{\min} + \alpha}{2} \right] &= \sin \left[\frac{\delta_{\min}}{2} \right] \cos \left[\frac{\alpha}{2} \right] + \cos \left[\frac{\delta_{\min}}{2} \right] \sin \left[\frac{\alpha}{2} \right] \\ \Rightarrow 1.5 \cdot \sin \left[\frac{\alpha}{2} \right] &= \sin \left[\frac{\delta_{\min}}{2} \right] \cos \left[\frac{\alpha}{2} \right] + \cos \left[\frac{\delta_{\min}}{2} \right] \sin \left[\frac{\alpha}{2} \right] \\ \Rightarrow \left(1.5 - \cos \left[\frac{\delta_{\min}}{2} \right] \right) \sin \left[\frac{\alpha}{2} \right] &= \sin \left[\frac{\delta_{\min}}{2} \right] \cos \left[\frac{\alpha}{2} \right] \\ \Rightarrow \frac{\sin \left[\frac{\alpha}{2} \right]}{\cos \left[\frac{\alpha}{2} \right]} = \tan \left[\frac{\alpha}{2} \right] &= \frac{\sin \left[\frac{\delta_{\min}}{2} \right]}{1.5 - \cos \left[\frac{\delta_{\min}}{2} \right]} \\ \Rightarrow \alpha = 2 \cdot \tan^{-1} \left[\frac{\sin \left[\frac{\delta_{\min}}{2} \right]}{1.5 - \cos \left[\frac{\delta_{\min}}{2} \right]} \right] \end{aligned}$$

If no ray “gets out” of the prism, the angle of deviation is $\delta_{\min} \geq 90^\circ = \frac{\pi}{2}$ radians:

$$\begin{aligned} \alpha_{\min} &= 2 \cdot \tan^{-1} \left[\frac{\sin \left[\frac{\left(\frac{\pi}{2}\right)}{2} \right]}{1.5 - \cos \left[\frac{\left(\frac{\pi}{2}\right)}{2} \right]} \right] = 2 \cdot \tan^{-1} \left[\frac{\sin \left[\frac{\pi}{4} \right]}{1.5 - \cos \left[\frac{\pi}{4} \right]} \right] \\ &= 1.4565 \text{ radians} \simeq 83 \text{ deg } 27 \text{ min } 12.7 \text{ sec} \end{aligned}$$

If $\alpha > \alpha_{\min}$, no ray escapes.