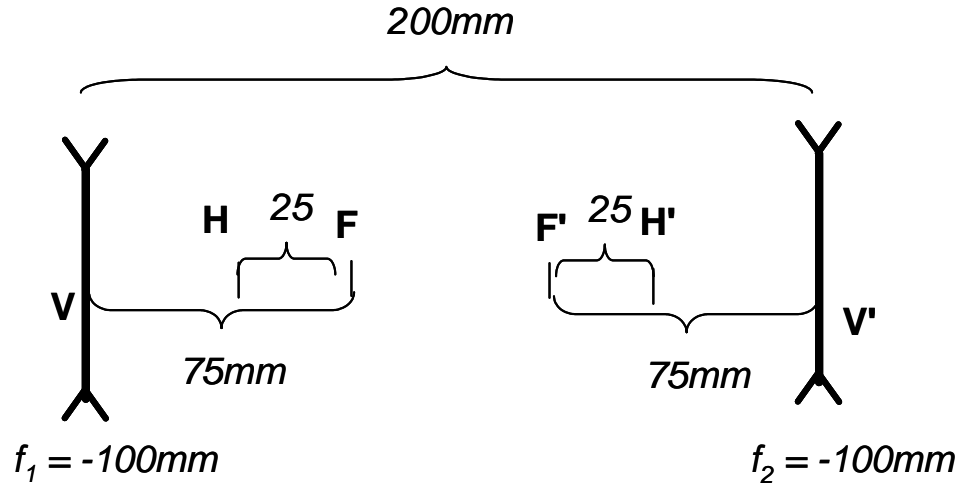


Do all problems – point totals are given. Staple problems together and submit *in numerical order*.

1. (30%) An imaging system is constructed of two identical thin lenses L_1 and L_2 with $f_1 = f_2 = -100$ mm and diameters of 25 mm. The lenses are separated by $t = +200$ mm and all are in air.

- (a) Sketch the system.



- (b) Determine the equivalent (effective) focal length of the system.

$$\begin{aligned}
 f_{eff} &= \overline{H'F'} = \overline{FH} \\
 &= \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2} \right)^{-1} = \frac{f_1 f_2}{f_1 + f_2 - t} \\
 &= \frac{(-100 \text{ mm})(-100 \text{ mm})}{-100 \text{ mm} + (-100 \text{ mm}) - 200 \text{ mm}} = \boxed{-25 \text{ mm} = f_{eff}}
 \end{aligned}$$

- (c) Find the focal and principal points of the system and locate them on the sketch of part (a).

$$\begin{aligned}
 BFD &= \overline{V'F'} = \frac{f_1 f_2 - f_2 t}{f_1 + f_2 - t} = \boxed{-75 \text{ mm} = \overline{V'F'}} \\
 \overline{H'V'} + \overline{V'F'} &= \overline{H'F'} \implies \overline{H'V'} = \overline{H'F'} - \overline{V'F'} = -25 \text{ mm} - (-75 \text{ mm}) \\
 &= \boxed{+50 \text{ mm} = \overline{H'V'}} \\
 FFD &= \overline{FV} = \frac{f_1 f_2 - f_1 t}{f_1 + f_2 - t} = -75 \text{ mm} \implies \boxed{\overline{FV} = +75 \text{ mm}} \\
 \overline{VH} + \overline{FV} &= \overline{FH} \implies \overline{VH} = \overline{FH} - \overline{FV} = -25 \text{ mm} - (-75 \text{ mm}) \\
 &= \boxed{+50 \text{ mm} = \overline{VH}}
 \end{aligned}$$

- (d) Determine which lens is the “aperture stop” of the system and explain your reasoning.
If the object is real, the lens L_2 constrains the cone of rays in the system, so it is the stop.

- (e) Locate the entrance and exit pupils of the system and find their sizes.

The entrance pupil is the image of the stop from object space, so it is the image of an object with diameter 25 mm at a distance of 200 mm seen through a negative lens with focal length $f_1 = 100$ mm. Use the thin-lens imaging equation to find the image location and magnification:

$$s' = \left(\frac{1}{f_1} - \frac{1}{t} \right)^{-1} = -\frac{200}{3} \text{ mm} = -66\frac{2}{3} \text{ mm}$$

$$M_T = -\frac{s'}{s} = -\frac{(-\frac{200}{3} \text{ mm})}{200 \text{ mm}} = +\frac{1}{3} \implies \text{Exit pupil size is } +\frac{1}{3} \cdot 25 \text{ mm} = +8\frac{1}{3} \text{ mm}$$

The exit pupil is the image of the stop seen from image space, and so coincides with the stop and its magnification is +1.

- (f) USE THE DISTANCES TO THE PRINCIPAL POINTS to find the image of an object located at the point \mathbf{O} such that with $\overline{\mathbf{OV}} = 100$ mm. Also find the transverse magnification of this image.

$$\overline{\mathbf{OV}} = +100 \text{ mm} \implies \overline{\mathbf{OH}} = s = \overline{\mathbf{OV}} + \overline{\mathbf{VH}} = +100 \text{ mm} + (+50 \text{ mm}) = +150 \text{ mm} = s$$

$$s' = \left(\frac{1}{f_{eff}} - \frac{1}{s} \right)^{-1} = \left(\frac{1}{-25 \text{ mm}} - \frac{1}{+150 \text{ mm}} \right)^{-1} = -\frac{150}{7} \text{ mm} \simeq -21.43 \text{ mm}$$

$$s' = \overline{\mathbf{H'O'}} = -\frac{150}{7} \text{ mm}$$

$$\overline{\mathbf{H'O'}} = \overline{\mathbf{H'V'}} + \overline{\mathbf{V'O'}} \implies \overline{\mathbf{V'O'}} = -\frac{150}{7} \text{ mm} - 50 \text{ mm}$$

$$= \boxed{-\frac{500}{7} \text{ mm} = \overline{\mathbf{V'O'}} \simeq -71.43 \text{ mm}} \implies \text{VIRTUAL IMAGE}$$

$$M_T = -\frac{s'}{s} = -\frac{-\frac{150}{7} \text{ mm}}{+150 \text{ mm}} = \boxed{M_T = +\frac{1}{7}}$$

- (g) USE THE “BRUTE-FORCE” METHOD (find the images created by each lens in sequence) to determine the location of the image of an object located at the point with $\overline{\mathbf{OV}} = 100$ mm; also find the transverse magnification of the image. These answers should confirm those of the previous question.

$$s'_1 = \left(\frac{1}{f_1} - \frac{1}{s_1} \right)^{-1} = \left(\frac{1}{-100 \text{ mm}} - \frac{1}{+100 \text{ mm}} \right)^{-1} = -50 \text{ mm}$$

$$(M_T)_1 = -\frac{s'_1}{s_1} = -\frac{-50 \text{ mm}}{+100 \text{ mm}} = +\frac{1}{2}$$

$$s_2 = t - s'_1 = 200 \text{ mm} - (-50 \text{ mm}) = +250 \text{ mm}$$

$$s'_2 = \left(\frac{1}{f_2} - \frac{1}{s_2} \right)^{-1} = \left(\frac{1}{-100 \text{ mm}} - \frac{1}{+250 \text{ mm}} \right)^{-1} = -\frac{500}{7} \text{ mm} \simeq -71.43 \text{ mm}$$

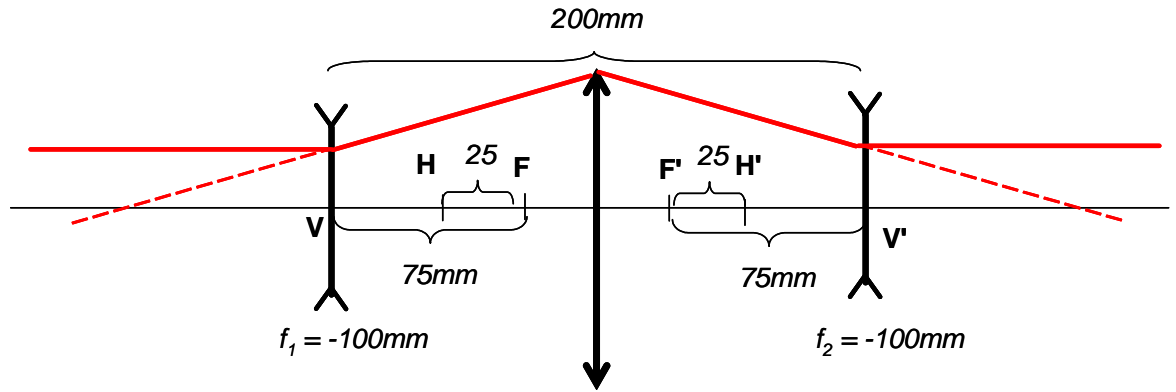
$$(M_T)_2 = -\frac{s'_2}{s_2} = -\frac{-\frac{500}{7} \text{ mm}}{+250 \text{ mm}} = +\frac{2}{7}$$

$$M_T = (M_T)_1 \cdot (M_T)_2 = -\frac{1}{2} \cdot +\frac{2}{7} = -\frac{1}{7}$$

- (h) Determine the longitudinal magnifications of the image of the object with $\overline{OV} = 100 \text{ mm}$

$$M_L = -(M_T)^2 = -\frac{1}{49}$$

- (i) Now add a third thin lens to the system that is placed in the center of the system (i.e., midway between the first two lenses) with focal length f_3 . Determine the focal length f_3 that creates a three-lens telescope and find its angular magnification.



From sketch, the lens at the center must create an image of an object located at the image-space focal point of lens L_1 and create an image at the object-space focal point of L_2 , so the object and image distances are both $+200 \text{ mm}$. Thus the focal length of the third lens must be $f_3 = +100 \text{ mm}$. The outgoing ray is at the same height as the incoming ray – in fact, the system acts as though it “isn’t there”, to the angular magnification is unity.

- (j) Find the focal length f_3 (between L_1 and L_2) such that the focal length of the optical system is $f_{eff} = f_3$.

I found the vertex-to-vertex matrix of the system and evaluated the focal length:

$$\begin{aligned} \mathcal{M}_{VV'} &= \mathcal{R}_3 \mathcal{T}_2 \mathcal{R}_2 \mathcal{T}_1 \mathcal{R}_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{-100 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{-100 \text{ mm}} & 1 \end{bmatrix} \\ &\quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -\frac{200}{f_3} + 3 & -\frac{10000}{f_3} + 200 \\ -\frac{4}{f_3} + \frac{1}{25} & -\frac{200}{f_3} + 3 \end{bmatrix} \end{aligned}$$

$$\text{The power of the system is } \varphi_{eff} = -C = \frac{4}{f_3} - \frac{1}{25} = +\frac{1}{f_3}$$

$$\Rightarrow \frac{1}{25} = \frac{3}{f_3} \Rightarrow \boxed{f_3 = +75 \text{ mm}}$$

Check it by plugging into matrices:

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ -\frac{1}{-100 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{75 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{-100 \text{ mm}} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{200}{3} \text{ mm} \\ -\frac{1}{75 \text{ mm}} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

So the focal lengths of the system is the same as f_3 .

- (k) (OPTIONAL BONUS) Find the focal length of the third lens (between L_1 and L_2) that generates an optical system where the two focal points coincide.

By symmetry, the two focal points will have to coincide at the center, i.e., at the location of the “third” lens in the middle. So we need to find the focal length such that the $BFD = FFD = -100$ mm.. The system (vertex-to-vertex matrix) is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -\frac{200}{f_3} + 3 & -\frac{10000}{f_3} + 200 \\ -\frac{4}{f_3} + \frac{1}{25} & -\frac{200}{f_3} + 3 \end{bmatrix}$$

We saw in the discussion of matrices that:

$$\begin{aligned} BFD &= \overline{\mathbf{V}'\mathbf{F}'} = -\frac{A}{C} = -\frac{\left(-\frac{200}{f_3} + 3\right)}{\left(-\frac{4}{f_3} + \frac{1}{25}\right)} = -100 \text{ mm} \\ -\frac{400}{f_3} + 4 &= -\frac{200}{f_3} + 3 \\ -\frac{200}{f_3} &= -1 \implies \boxed{f_3 = +200 \text{ mm}} \end{aligned}$$

Check it by inserting into matrix:

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ +\frac{1}{100 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{200 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ +\frac{1}{100 \text{ mm}} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 150 \text{ mm} \\ \frac{1}{50 \text{ mm}} & 2 \end{bmatrix} \\ &f_{eff} = -\frac{1}{C} = -50 \text{ mm} \\ BFD &= \overline{\mathbf{V}'\mathbf{F}'} = -\frac{A}{C} = -\frac{2}{\frac{1}{50 \text{ mm}}} = -100 \text{ mm} \\ FFD &= \overline{\mathbf{F}\mathbf{V}} = -\frac{D}{C} = -\frac{2}{\frac{1}{50 \text{ mm}}} = -100 \text{ mm} \end{aligned}$$

- (l) (OPTIONAL BONUS) Find the focal length of the third lens (between L_1 and L_2) that generates an optical system where the two principal points coincide.

Again, by symmetry, we might expect the two principal points to coincide at the center, so that $\overline{\mathbf{V}\mathbf{H}} = \overline{\mathbf{H}'\mathbf{V}'} = +100$ mm. We know from the matrix formulation that $f_{eff} = \overline{\mathbf{F}\mathbf{H}} = \overline{\mathbf{H}'\mathbf{F}'} = -\frac{1}{C}$ and that the BFD and FFD are given above. We can combine these to get $\overline{\mathbf{V}\mathbf{H}}$ and $\overline{\mathbf{H}'\mathbf{V}'}$:

$$\begin{aligned}
\overline{\mathbf{H}'\mathbf{V}'} + \overline{\mathbf{V}'\mathbf{F}'} &= \overline{\mathbf{H}'\mathbf{F}'} \\
\Rightarrow \overline{\mathbf{H}'\mathbf{V}'} &= \overline{\mathbf{H}'\mathbf{F}'} - \overline{\mathbf{V}'\mathbf{F}'} \\
&= -\frac{1}{C} - \left(-\frac{A}{C}\right) = \frac{A-1}{C} = -100 \text{ mm} \\
\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} -\frac{200}{f_3} + 3 & -\frac{10000}{f_3} + 200 \\ -\frac{4}{f_3} + \frac{1}{25} & -\frac{200}{f_3} + 3 \end{bmatrix} \\
\Rightarrow \frac{A-1}{C} &= \frac{-\frac{200}{f_3} + 2}{-\frac{4}{f_3} + \frac{1}{25}} = -100 \text{ mm} \\
-200 + 2f_3 &= 400 - 4f_3 \Rightarrow \boxed{f_3 = +100 \text{ mm}}
\end{aligned}$$

Check it by inserting into matrix:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ +\frac{1}{100 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{100 \text{ mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ +\frac{1}{100 \text{ mm}} & 1 \end{bmatrix} \\
= \begin{bmatrix} 1 & 100 \text{ mm} \\ 0 & 1 \end{bmatrix} \\
f_{eff} = -\frac{1}{C} = \infty \Rightarrow \text{telescope}
\end{aligned}$$

so the principal points actually are at ∞ !

2. (20%) For ONE of your eyes:

- (a) Use a ruler to find the APPROXIMATE distance to the closest point where you see a “clearly focused” image (this is called the “near point” of the eye). The generally accepted distance between the eye and the near point is 10 in = 254 mm, but use your own measurement in the following sections.

For my eye, the distance is approximately 6 in, so I'll call it 150 mm

- (b) Determine the approximate angle that a Lincoln penny would subtend at this distance for your eye and compare it to the angular subtense for a Lincoln penny viewed at the “standard” distance for the near point. Is your eye better or less suited than the “standard” eye for performing work that requires vision of fine detail?

Diameter of Lincoln penny is approximately 19 mm. At a distance of $\ell = 150$ mm, the angular subtense is:

$$\begin{aligned}(\Delta\theta)_{me} &= \frac{d}{\ell} = \frac{19 \text{ mm}}{150 \text{ mm}} \simeq 0.127 \text{ radians} \simeq 7.3^\circ \\(\Delta\theta)_{standard} &= \frac{19 \text{ mm}}{254 \text{ mm}} \simeq 0.075 \text{ radians} \simeq 4.3^\circ\end{aligned}$$

So the penny subtends a larger angle when seen by my eye, which means that I can see fine detail better than the standard viewer.

- (c) You are now to add an additional thin lens with focal length f that will be held between the eye and the penny to create an image of the penny at the near point of your eye. Determine the location of the lens and of the penny that produces such an image and find its transverse magnification.

If I hold a positive lens close to my eye and hold the penny so that the image is at the near point of my eye, then I can find the distance from the object to the lens. I assumed that the focal length is $f = +50$ mm

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \text{ where } s' = 150 \text{ mm}$$

$$\text{If } f = 50 \text{ mm, then } s = \left(\frac{1}{f} - \frac{1}{s'}\right)^{-1} = \left(\frac{1}{50 \text{ mm}} - \frac{1}{150 \text{ mm}}\right)^{-1} = +75 \text{ mm}$$

$$M_T = -\frac{s'}{s} = -\frac{150 \text{ mm}}{75 \text{ mm}} = +2$$

$$\text{If } f = 25 \text{ mm, then } s = \left(\frac{1}{25 \text{ mm}} - \frac{1}{150 \text{ mm}}\right)^{-1} = +30 \text{ mm}$$

$$M_T = -\frac{s'}{s} = -\frac{150 \text{ mm}}{30 \text{ mm}} = +5$$

This is a magnifier, or magnifying glass.

3. (20%) During the term, we have seen several examples of the “superposition” of sinusoidal waves with different spatial or temporal periods. For example, we showed that:

$$\begin{aligned}\cos[\omega_0 t] + \cos[\omega_1 t] &= 2 \cos\left[\frac{\omega_0 + \omega_1}{2}t\right] \cos\left[\frac{\omega_0 - \omega_1}{2}t\right] \\ &\equiv 2 \cos[\omega_{avg}t] \cos[\omega_{mod}t]\end{aligned}$$

and extended this result to three dimensional waves to evaluate the pattern that results from two apertures via:

$$\cos[\mathbf{k}_0 \bullet \mathbf{r} - \omega_0 t] + \cos[\mathbf{k}_1 \bullet \mathbf{r} - \omega_1 t] = 2 \cos[\mathbf{k}_{avg} \bullet \mathbf{r} - \omega_{avg} t] \cdot \cos[\mathbf{k}_{mod} \bullet \mathbf{r} - \omega_{mod} t]$$

where the scalar product of two vectors is defined:

$$\mathbf{k} \bullet \mathbf{r} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k_x x + k_y y + k_z z$$

and the length of the vector $\mathbf{k} = \sqrt{k_x^2 + k_y^2 + k_z^2}$. In all of the the following, assume that $(k_x)_0 = (k_x)_1 = 0$.

- (a) Find an expression for and sketch the output amplitude and irradiance (intensity) if $(k_y)_0 = -(k_y)_1$ and $(k_z)_0 = (k_z)_1$.

This is exactly the two-slit problem:

$$\begin{aligned}\mathbf{k}_{avg} &= \frac{\mathbf{k}_0 + \mathbf{k}_1}{2} = \hat{\mathbf{z}}k_z = \hat{\mathbf{z}} \cdot \frac{2\pi}{\lambda} \cos[\theta] \\ \mathbf{k}_{mod} &= \frac{\mathbf{k}_0 - \mathbf{k}_1}{2} = \hat{\mathbf{y}}k_y = \hat{\mathbf{y}} \cdot \frac{2\pi}{\lambda} \sin[\theta] \\ |\mathbf{k}_0| &= |\mathbf{k}_1| = \frac{2\pi}{\lambda} \implies \omega_0 = \omega_1 \implies \omega_{avg} = \omega_0, \omega_{mod} = 0\end{aligned}$$

Amplitude:

$$\begin{aligned}f[x, y, z, t] &= 2 \cos[\mathbf{k}_{avg} \bullet \mathbf{r} - \omega_{avg} t] \cdot \cos[\mathbf{k}_{mod} \bullet \mathbf{r} - \omega_{mod} t] \\ &= 2 \cos\left[\frac{2\pi z}{\lambda} \cos[\theta] - 2\pi\nu_1 t\right] \cdot \cos\left[\frac{2\pi}{\lambda} y \sin[\theta]\right]\end{aligned}$$

Irradiance:

$$\begin{aligned}\langle |f[x, y, z, t]|^2 \rangle &= 4 \left\langle \cos^2\left[\frac{2\pi z}{\lambda} \cos[\theta] - 2\pi\nu_1 t\right] \right\rangle \cdot \cos^2\left[\frac{2\pi}{\lambda} y \sin[\theta]\right] \\ &= 4 \cdot \frac{1}{2} \cdot \cos^2\left[2\pi \frac{y}{\left(\frac{\lambda}{\sin[\theta]}\right)}\right] = 2 \cdot \frac{1}{2} \left(1 + \cos\left(2\pi \frac{y}{\left(\frac{\lambda}{2 \sin[\theta]}\right)}\right)\right)\end{aligned}$$

Period:

$$D = \frac{\lambda}{2 \sin[\theta]}$$

- (b) Find an expression for and sketch the output amplitude and irradiance (intensity) if $(k_y)_0 = -(k_y)_1$ and $(k_z)_0 = (k_z)_1$ AND if the relative phase of the two light sources is π radians.

This is the two-slit problem with a phase change:

$$\begin{aligned} & \cos [\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t + \phi_0] + \cos [\underline{\mathbf{k}}_1 \bullet \underline{\mathbf{r}} - \omega_1 t + \phi_1] \\ &= \cos [\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t + \phi_0] + \cos [\underline{\mathbf{k}}_1 \bullet \underline{\mathbf{r}} - \omega_1 t + (\phi_0 + \pi)] \\ &= 2 \cos [\underline{\mathbf{k}}_{avg} \bullet \underline{\mathbf{r}} - \omega_{avg} t + \phi_{avg}] \cdot \cos [\underline{\mathbf{k}}_{mod} \bullet \underline{\mathbf{r}} - \omega_{mod} t + \phi_{mod}] \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{k}}_{avg} &= \hat{\mathbf{z}} \cdot \frac{2\pi}{\lambda} \cos [\theta] \\ \underline{\mathbf{k}}_{mod} &= \hat{\mathbf{y}} \cdot \frac{2\pi}{\lambda} \sin [\theta] \\ |\underline{\mathbf{k}}_1| &= |\underline{\mathbf{k}}_2| = \frac{2\pi}{\lambda} \implies \omega_1 = \omega_2 \implies \omega_{avg} = \omega_1, \omega_{mod} = 0 \\ \phi_{avg} &= \frac{\phi_0 + \phi_1}{2} = \frac{\phi_0 + (\phi_0 + \pi)}{2} = \phi_0 + \frac{\pi}{2} \\ \phi_{mod} &= \frac{\phi_0 - \phi_1}{2} = \frac{\phi_0 - (\phi_0 + \pi)}{2} = -\frac{\pi}{2} \end{aligned}$$

Amplitude:

$$\begin{aligned} f [x, y, z, t] &= 2 \cos \left[\underline{\mathbf{k}}_{avg} \bullet \underline{\mathbf{r}} - \omega_{avg} t + \left(\phi_0 + \frac{\pi}{2} \right) \right] \cdot \cos [\underline{\mathbf{k}}_{mod} \bullet \underline{\mathbf{r}} - \omega_{mod} t + \phi_{mod}] \\ &= 2 \cos \left[\frac{2\pi z}{\lambda_0} \cos [\theta] - 2\pi \nu_0 t + \left(\phi_0 + \frac{\pi}{2} \right) \right] \cdot \cos \left[\frac{2\pi}{\lambda_0} y \sin [\theta] - \frac{\pi}{2} \right] \end{aligned}$$

Irradiance:

$$\begin{aligned} \langle |f [x, y, z, t]|^2 \rangle &= 4 \left\langle \cos^2 \left[\frac{2\pi z}{\lambda} \cos [\theta] - 2\pi \nu_0 t + \left(\phi_0 + \frac{\pi}{2} \right) \right] \right\rangle \cdot \left(\cos \left[\frac{2\pi}{\lambda_0} y \sin [\theta] - \frac{\pi}{2} \right] \right)^2 \\ &= 4 \left\langle \cos^2 \left[\frac{2\pi z}{\lambda_0} \cos [\theta] - 2\pi \nu_0 t + \left(\phi_0 + \frac{\pi}{2} \right) \right] \right\rangle \cdot \left(\sin \left[\frac{2\pi}{\lambda_0} y \sin [\theta] \right] \right)^2 \\ &= 4 \cdot \frac{1}{2} \cdot \sin^2 \left[2\pi \frac{y}{\left(\frac{\lambda_0}{\sin[\theta]} \right)} \right] = 2 \cdot \frac{1}{2} \left(1 - \cos \left(2\pi \frac{y}{\left(\frac{\lambda_0}{2 \sin[\theta]} \right)} \right) \right) \end{aligned}$$

Period is unchanged, but the sinusoidal fringe pattern has a null at the center instead of a maximum:

- (c) Find an expression for and sketch the output amplitude and irradiance (intensity) if $(k_y)_0 = (k_y)_1 = 0$ and $(k_z)_0 = -(k_z)_1$.

$$\begin{aligned} \underline{\mathbf{k}}_{avg} &= \frac{\underline{\mathbf{k}}_0 + \underline{\mathbf{k}}_1}{2} = \mathbf{0} \\ \underline{\mathbf{k}}_{mod} &= \frac{\underline{\mathbf{k}}_0 - \underline{\mathbf{k}}_1}{2} = \hat{\mathbf{z}} k_z \\ |\underline{\mathbf{k}}_0| &= |\underline{\mathbf{k}}_1| = \frac{2\pi}{\lambda} \implies \omega_0 = \omega_1 \implies \omega_{avg} = \omega_0, \omega_{mod} = 0 \end{aligned}$$

Amplitude:

$$\begin{aligned}
f[x, y, z, t] &= 2 \cos [\underline{\mathbf{k}}_{avg} \bullet \underline{\mathbf{r}} - \omega_{avg} t] \cdot \cos [\underline{\mathbf{k}}_{mod} \bullet \underline{\mathbf{r}} - \omega_{mod} t] \\
&= 2 \cos [0 - 2\pi\nu_0 t] \cdot \cos \left[\frac{2\pi}{\lambda_0} z \right] \\
&= 2 \cos [2\pi\nu_0 t] \cos \left[\frac{2\pi}{\lambda_0} z \right] \implies \text{standing wave}
\end{aligned}$$

Irradiance:

$$\begin{aligned}
\langle |f[x, y, z, t]|^2 \rangle &= 4 \langle \cos^2 [2\pi\nu_0 t] \rangle \cdot \cos^2 \left[\frac{2\pi}{\lambda_0} z \right] \\
&= 4 \cdot \frac{1}{2} \cdot \cos^2 \left[2\pi \frac{y}{\lambda_0} \right] = 2 \cdot \frac{1}{2} \left(1 + \cos \left(2\pi \frac{y}{\left(\frac{\lambda_0}{2} \right)} \right) \right)
\end{aligned}$$

Period:

$$D = \frac{\lambda_0}{2}$$

- (d) Find an expression for and sketch the output amplitude and irradiance (intensity) if $(k_y)_0 = -(k_y)_1$ and $|\underline{\mathbf{k}}_0| \neq |\underline{\mathbf{k}}_1|$.

The light is traveling at the same angle as before, but the wavelengths and temporal frequencies are different

$$\begin{aligned}
\omega_{avg} &= \frac{\omega_0 + \omega_1}{2} \\
\omega_{mod} &= \frac{\omega_0 - \omega_1}{2}
\end{aligned}$$

and the average and modulation wavevectors are:

$$\begin{aligned}
\underline{\mathbf{k}}_{avg} &= \frac{\underline{\mathbf{k}}_0 + \underline{\mathbf{k}}_1}{2} = \hat{\underline{\mathbf{z}}} \frac{(k_z)_0 + (k_z)_1}{2} \\
\underline{\mathbf{k}}_{mod} &= \frac{\underline{\mathbf{k}}_0 - \underline{\mathbf{k}}_1}{2} = \hat{\underline{\mathbf{z}}} \frac{(k_z)_0 - (k_z)_1}{2} + \hat{\underline{\mathbf{y}}} (k_y)_0
\end{aligned}$$

Amplitude:

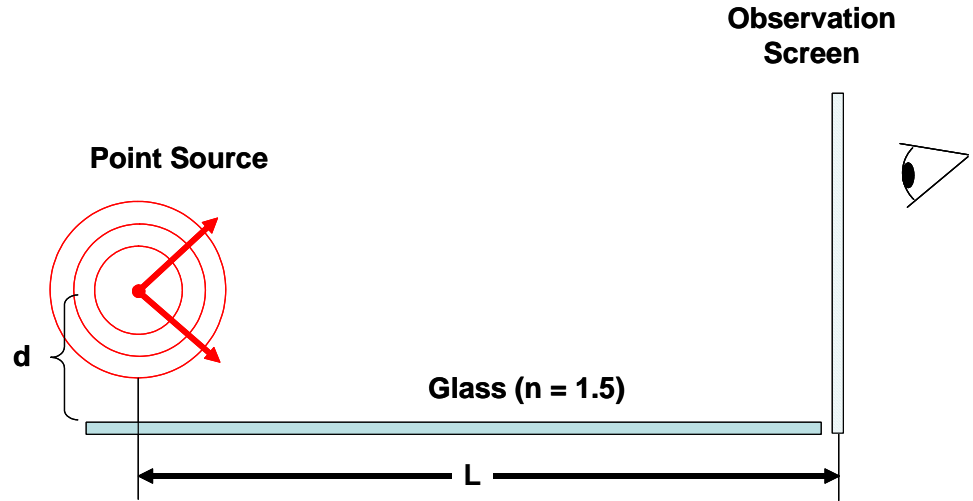
$$\begin{aligned}
f[x, y, z, t] &= 2 \cos [\underline{\mathbf{k}}_{avg} \bullet \underline{\mathbf{r}} - \omega_{avg} t] \cdot \cos [\underline{\mathbf{k}}_{mod} \bullet \underline{\mathbf{r}} - \omega_{mod} t] \\
&= 2 \cos \left[\left(\frac{(k_z)_0 + (k_z)_1}{2} \right) z - \left(\frac{\omega_0 + \omega_1}{2} \right) t \right] \\
&\quad \cdot \cos \left[\left((k_y)_0 y + \frac{(k_z)_0 - (k_z)_1}{2} \right) z - \left(\frac{\omega_0 - \omega_1}{2} \right) t \right]
\end{aligned}$$

Irradiance:

$$\begin{aligned}
\langle |f[x, y, z, t]|^2 \rangle &= 4 \left\langle \cos^2 \left[\left(\frac{(k_z)_0 + (k_z)_1}{2} \right) z - \left(\frac{\omega_0 + \omega_1}{2} \right) t \right] \right\rangle \\
&\quad \cdot \left\langle \cos^2 \left[\left((k_y)_0 y + \frac{(k_z)_0 - (k_z)_1}{2} \right) z - \left(\frac{\omega_0 - \omega_1}{2} \right) t \right] \right\rangle \\
&= 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1
\end{aligned}$$

No fringes

4. (20%) Consider the system shown below that consists of a point source of light with wavelength $\lambda_0 = 632.8 \text{ nm}$ that is located a distance d above the reflecting surface of a sheet of glass with $n = 1.5$. The observation screen has a diffusing surface that allows the eye to view the pattern of light from the source. The distance L from the point source to the observation screen is “large”.



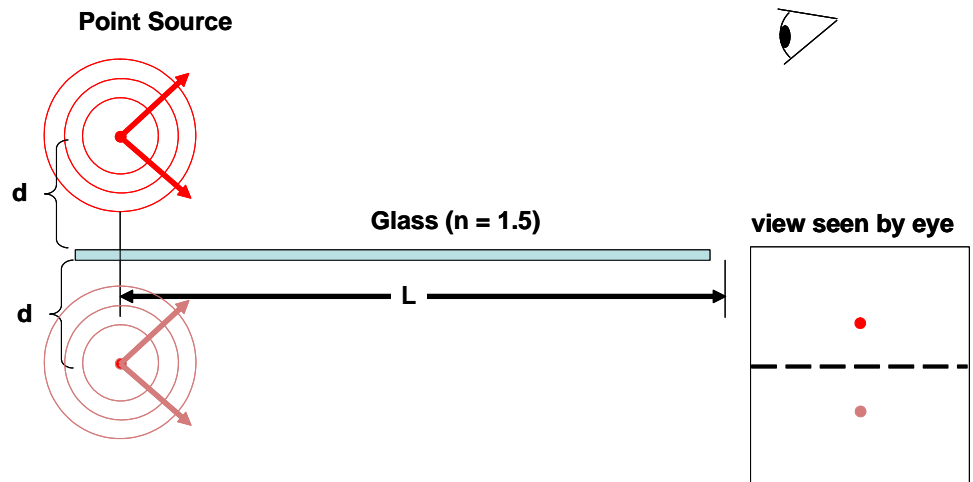
- (a) Describe what the eye sees if the observation screen is removed and the eye is focused on the point source.

The light that is reflected from the glass creates a virtual image of the source located d units BELOW the glass, so the separation between the real and virtual sources is $2d$ (NOT d , as we're used to), as shown. Note that the light reflected from the glass is reduced in intensity by the factor of the reflectance from the glass. In this case, I will just use the value for normal incidence:

$$\text{amplitude reflectance} : \rho = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2$$

$$\text{intensity reflectance} : R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left(\frac{1 - 1.5}{1 + 1.5} \right)^2 = +0.04$$

Thus only 4% of the light is reflected, AND its phase is changed by π radians due to the negative sign in ρ .



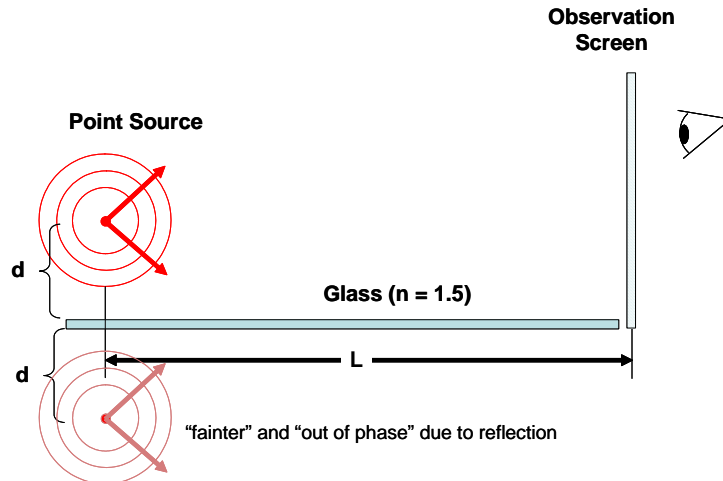
- (b) Describe and sketch what the eye sees on the observation screen. Be as quantitative as possible.

This is a variation of two-source interference. The light from the virtual source adds to the light viewed directly to create interference fringes. However, there are (at least) two other issues to consider: the reduction in amplitude due to the reflectance and the phase change. You also might consider any polarization effects. The lower amplitude of the reflected light means that we have to divide the light from the real source into two parts: with an amplitude of 0.8 and with an amplitude of 0.2, where the latter interferes with the light from the virtual source, while the former just creates a bright field of light. The period of the fringes is easy to obtain from the formula where we have to remember that the separation between the sources is $2d$ instead of d :

$$L\lambda = (2d) D \implies D = \lambda \left(\frac{L}{2d} \right)$$

The phase change of the light creates a “null” fringe at the glass. This is EXACTLY like the result of problem 3b above.

$$\begin{aligned} f[y, z, t] &= 0.8 \cos[k_y y + k_z z - \omega_0 t] + (0.2 \cos[k_y y + k_z z - \omega_0 t] - 0.2 \cos[-k_y y + k_z z - \omega_0 t]) \\ &= 0.8 \cos[k_y y + k_z z - \omega_0 t] + 0.2 \cdot (2 \cdot \cos[k_z z - \omega_0 t] \cdot \cos[k_y y]) \\ \langle |f[y, z, t]|^2 \rangle &= \langle |0.8 \cos[k_y y + k_z z - \omega_0 t] + 0.2 \cdot (-2 \cdot \sin[k_z z - \omega_0 t] \cdot \sin[k_y y])|^2 \rangle \\ &\simeq 0.64 \langle \cos^2[k_y y + k_z z - \omega_0 t] \rangle + 0.16 \langle \sin^2[k_z z - \omega_0 t] \rangle \sin^2[k_y y] \\ &= 0.64 \cdot \frac{1}{2} + 0.08 \sin^2[k_y y] \\ &= 0.32 + 0.08 \left(\frac{1}{2} - \frac{1}{2} \cos \left[2\pi \frac{y}{D} \right] \right) \text{ where } D = \lambda \left(\frac{L}{2d} \right) \end{aligned}$$



- (c) What happens to the observed light if the “height” d of the source above the mirror is varied (increased or decreased).

Just as before, the period D of the fringes decreases if d increases, and vice versa.

5. (10%) Sometimes exams don't cover the aspects of the subject that you studied most carefully or in which you have the greatest interest. For that reason, I offer this "free" question; write your own problem or describe an optical system, effect, or theorem of your choice that we studied and describe its relevance to imaging (excluding those already included in the problems above). For example, you could consider the phenomenon of diffraction and its effect on an imaging system. You will be judged on both the suitability and caliber of the question and of the answer.

See individual solutions on your paper.