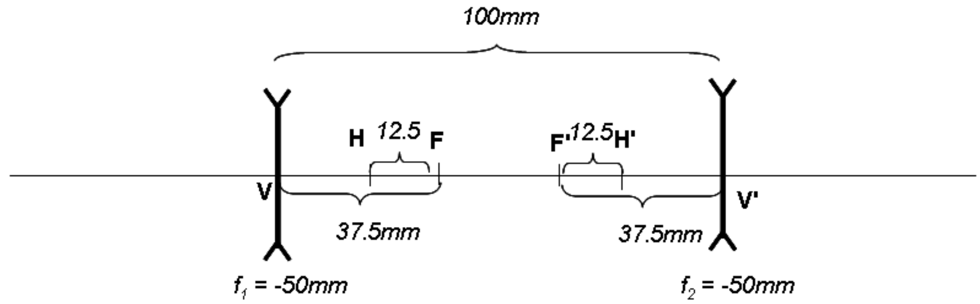
SIMG-303-20033 IN-CLASS FINAL EXAM

Do all problems – point totals are given. Staple problems together and submit *in numerical order.*

1. (40%) An imaging system is constructed of two identical thin lenses L_1 and L_2 with $f_1 = f_2 = -50\,\text{mm}$. The lenses are separated by $t = +100\,\text{mm}$ and everything is in air.

   (a) Sketch the system.



   (b) Determine the "equivalent" ("effective") focal length of the system.

$$
\begin{aligned}
f_{eff} &= \overline{\mathbf{H'F'}} = \overline{\mathbf{FH}} \\
&= \left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}\right)^{-1} = \frac{f_1 f_2}{f_1 + f_2 - t} = \frac{(-50\,\text{mm})\,(-50\,\text{mm})}{-50\,\text{mm} + (-50\,\text{mm}) - 100\,\text{mm}} \\
&= -\frac{25}{2}\,\text{mm} = \boxed{-12.5\,\text{mm} = f_{eff}}
\end{aligned}
$$

   (c) Find the focal and principal points of the system and locate them on the sketch of part (a).

$$
\begin{aligned}
BFD &= \overline{\mathbf{V'F'}} = \frac{f_1 f_2 - f_2 t}{f_1 + f_2 - t} = -\frac{75}{2}\,\text{mm} = -37.5\,\text{mm} \\
\overline{\mathbf{H'V'}} + \overline{\mathbf{V'F'}} &= \overline{\mathbf{H'F'}} \Longrightarrow \overline{\mathbf{H'V'}} = \overline{\mathbf{H'F'}} - \overline{\mathbf{V'F'}} = -12.5\,\text{mm} - (-37.5\,\text{mm}) = +25\,\text{mm} \\
FFD &= \overline{\mathbf{FV}} = \frac{f_1 f_2 - f_1 t}{f_1 + f_2 - t} = -\frac{75}{2}\,\text{mm} = -37.5\,\text{mm} \\
\overline{\mathbf{VH}} + \overline{\mathbf{FV}} &= \overline{\mathbf{FH}} \Longrightarrow \overline{\mathbf{VH}} = \overline{\mathbf{FH}} - \overline{\mathbf{FV}} = -12.5\,\text{mm} - (-37.5\,\text{mm}) = +25\,\text{mm}
\end{aligned}
$$

   (d) Determine the position and transverse magnification of the image of an object located at the point $\mathbf{O}$ such that with $\overline{\mathbf{OV}} = +50\,\text{mm}$.

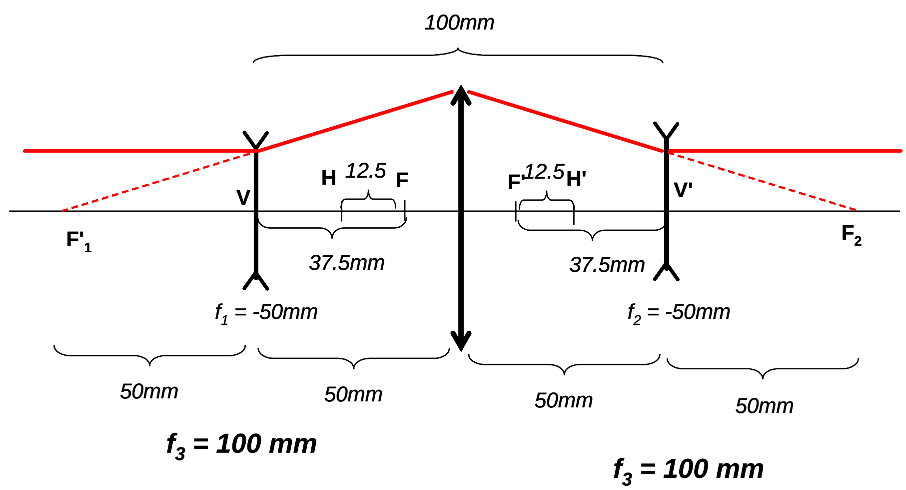
   By principal points:

$$
\begin{aligned}
\overline{\mathbf{OV}} &= +50\,\text{mm} \Longrightarrow \overline{\mathbf{OH}} = \overline{\mathbf{OV}} + \overline{\mathbf{VH}} = \overline{\mathbf{OV}} - \overline{\mathbf{HV}} = +50\,\text{mm} - (25\,\text{mm}) = +75\,\text{mm} = s \\
s' &= \overline{\mathbf{H'O'}} = \left(\frac{1}{f_{eff}} - \frac{1}{s}\right)^{-1} = \left(\frac{1}{-\frac{25}{2}\,\text{mm}} - \frac{1}{+75\,\text{mm}}\right)^{-1} = -\frac{75}{7}\,\text{mm} \simeq -10.714\,\text{mm} \\
\overline{\mathbf{V'O'}} &= \overline{\mathbf{H'O'}} - \overline{\mathbf{H'V'}} = -\frac{75}{7}\,\text{mm} - 25\,\text{mm} = -\frac{250}{7}\,\text{mm} \\
&= \boxed{s_2' \simeq -35.714\,\text{mm}} \Longrightarrow \text{VIRTUAL IMAGE} \\
M_T &= -\frac{s'}{s} = -\frac{-\frac{75}{7}\,\text{mm}}{75\,\text{mm}} = \boxed{M_T = \tfrac{1}{7}}
\end{aligned}
$$

By "brute force":

$$
\begin{aligned}
f_1 &= -50\,\text{mm} \\
s_1 &= +50\,\text{mm} \\
s_1' &= \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = \left(\frac{1}{-50\,\text{mm}} - \frac{1}{50\,\text{mm}}\right)^{-1} = -25\,\text{mm} \\
(M_T)_1 &= -\frac{s_1'}{s_1} = -\frac{-25\,\text{mm}}{50\,\text{mm}} = +\frac{1}{2} \\
s_2 &= t - s_1' = 100\,\text{mm} - (-25\,\text{mm}) = 125\,\text{mm} \\
s_2' &= \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = \left(\frac{1}{-50\,\text{mm}} - \frac{1}{125\,\text{mm}}\right)^{-1} = -\frac{250}{7}\,\text{mm} = \boxed{s_2' \simeq -35.714\,\text{mm}} \\
(M_T)_2 &= -\frac{s_2'}{s_2} = -\frac{-\frac{250}{7}\,\text{mm}}{125\,\text{mm}} = +\frac{2}{7} \\
M_T &= (M_T)_1 \cdot (M_T)_2 = +\frac{1}{2} \cdot +\frac{2}{7} = \boxed{M_T = \tfrac{1}{7}}
\end{aligned}
$$

(e) Now add a third thin lens to the system that is placed in the center of the system (i.e., midway between the first two lenses) with focal length f_3. Determine the focal length f_3 that creates a "three-lens telescope".



From sketch, the lens at the center must create an image of an object located at the image-space focal point of lens L_1 and create an image at the object-space focal point of L_2, so the object and image distances are both $+100\,\text{mm}$. Thus the focal length of the third lens must be $f_3 = +50\,\text{mm}$. The outgoing ray is at the same height as the incoming ray – in fact, the system acts as though it "isn't there", to the angular magnification is unity.

(f) OPTIONAL BONUS: Evaluate the effective focal length of the system if the lens with the focal length you derived in part (e) is placed between L_1 and L_2 a distance ℓ "behind" the first lens and a distance $t - \ell = 100\,\text{mm} - \ell$ "in front" of the second lens. Your expression for f_{eff} will be a function of ℓ.

$$\textit{I used matrices:}\mathcal{M}_{VV'} = \mathcal{R}_3\mathcal{T}_2\mathcal{R}_2\mathcal{T}_1\mathcal{R}_1$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{-50\,\text{mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 100\,\text{mm} - \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{50\,\text{mm}} & 1 \end{bmatrix} \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{-50\,\text{mm}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2500}\ell^2 - \frac{1}{50}\ell + +1 & \frac{1}{50}\ell^2 - 2\ell + +100 \\ \frac{1}{25}\left(\frac{1}{2500}\ell^2 - \frac{1}{25}\ell + 1\right) & \frac{1}{2500}\ell^2 - \frac{3}{50}\ell + 3 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\textit{The power of the system is } -C = -\frac{1}{25}\left(\frac{1}{2500}\ell^2 - \frac{1}{25}\ell + 1\right)$$

$$f_{eff} = -\frac{1}{C} = 25 \cdot \left(\frac{1}{2500}\ell^2 - \frac{1}{25}\ell + 1\right)^{-1}$$

$$, \textit{ if } \ell = 50\,\text{mm}, \textit{ then } \varphi_{eff} = 0 \Longrightarrow f_{eff} = \infty, \textit{ as expected}$$

2. (20%) We derived an expression for the sum of two oscillations with the same amplitude and different oscillation frequencies:

$$\cos[\omega_0 t] + \cos[\omega_1 t] = 2 \cdot \cos[\omega_{\text{mod}} t] \cdot \cos[\omega_{avg} t]$$

Extend this concept to consider the sum of two three-dimensional nondispersive TRAVELING plane waves that have unit amplitude. The first wave travels in the direction specified by:

$$\underline{\mathbf{k}}_1 = \begin{bmatrix} (k_x)_1 \\ (k_y)_1 \\ (k_z)_1 \end{bmatrix}, \text{ where } |\underline{\mathbf{k}}_1| = \sqrt{(k_x)_1^2 + (k_y)_1^2 + (k_y)_1^2} = \frac{2\pi}{\lambda_1}$$

where $(k_x)_1, (k_y)_1, (k_z)_1$ are respectively the x, y, and z compnents of the first wave vector The second plane wave travels in the direction (NOTE INDICES)

$$\underline{\mathbf{k}}_2 = \begin{bmatrix} (k_x)_2 \\ (k_y)_2 \\ (k_z)_2 \end{bmatrix} = \begin{bmatrix} -(k_x)_1 \\ -(k_y)_1 \\ (k_z)_2 \end{bmatrix}, \text{ where } |\underline{\mathbf{k}}_2| = \frac{2\pi}{\lambda_2}$$

$$f[x,y,z,t] = \cos[\underline{\mathbf{k}}_1 \bullet \underline{\mathbf{r}} - \omega_1 t + \phi_1] + \cos[\underline{\mathbf{k}}_2 \bullet \underline{\mathbf{r}} - \omega_2 t + \phi_2]$$
$$= 2 \cdot \cos\left[\left(\frac{\underline{\mathbf{k}}_1 + \underline{\mathbf{k}}_2}{2}\right) \bullet \underline{\mathbf{r}} - \left(\frac{\omega_1 + \omega_2}{2}\right) t + \left(\frac{\phi_1 + \phi_2}{2}\right)\right]$$
$$\cdot \cos\left[\left(\frac{\underline{\mathbf{k}}_1 - \underline{\mathbf{k}}_2}{2}\right) \bullet \underline{\mathbf{r}} - \left(\frac{\omega_1 - \omega_2}{2}\right) t + \left(\frac{\phi_1 - \phi_2}{2}\right)\right]$$

(a) If $\lambda_1 = \lambda_2$, find an expression for the amplitude AND the irradiance (intensity) of the resulting electric field viewed at some distance down the z axis towards $z = +\infty$.

$$\textit{if } \lambda_1 = \lambda_2, \textit{ then } |\underline{\mathbf{k}}_1| = |\underline{\mathbf{k}}_2| = \sqrt{(k_x)_2^2 + (k_y)_2^2 + (k_y)_2^2}$$
$$= \sqrt{(-(k_x)_1)^2 + (-(k_y)_1)^2 + (k_z)_2^2}$$
$$= \sqrt{((k_x)_1)^2 + ((k_y)_1)^2 + (k_z)_1^2} \Longrightarrow (k_z)_1 = (k_z)_2 = \frac{2\pi}{\lambda_1}\cos[\theta]$$
$$(k_y)_2 = -(k_y)_1 = \frac{2\pi}{\lambda_1}\sin[\theta]$$

$$\begin{aligned}
\textit{Amplitude} \quad &: \quad f[x,y,z,t] = 2\cos\left[\frac{2\pi z}{\lambda_1}\cos[\theta] - \omega_1 t\right]\cdot\cos\left[\frac{2\pi y}{\lambda_1}\sin[\theta]\right] \\
\textit{Irradiance} \quad &: \quad \left\langle |f[x,y,z,t]|^2\right\rangle = 4\left\langle \cos^2\left[\frac{2\pi z}{\lambda_1}\cos[\theta] - \omega_1 t\right]\right\rangle\cdot\cos^2\left[\frac{2\pi y}{\lambda_1}\sin[\theta]\right] \\
&= \quad 1+\cos\left[2\pi\frac{y}{\frac{\lambda_1}{2\sin[\theta]}}\right]
\end{aligned}$$

(b) Describe the QUALITATIVE difference that will result if $\lambda_1 \neq \lambda_2$.

The modulation wave also "travels" and so its irradiance averages to $\frac{1}{2}$; *there are no fringes.*

(c) OPTIONAL BONUS: Find the QUANTITATIVE expression for the sum of the two traveling waves in part (b). *Amplitude:*

$$\begin{aligned}
f[x,y,z,t] &= 2\ \cos\left[\underline{\mathbf{k}}_{avg}\bullet\underline{\mathbf{r}} - \omega_{avg}t\right]\cdot\cos\left[\underline{\mathbf{k}}_{\text{mod}}\bullet\underline{\mathbf{r}} - \omega_{\text{mod}}t\right] \\
&= 2\ \cos\left[\left(\frac{(k_z)_0+(k_z)_1}{2}\right)z - \left(\frac{\omega_0+\omega_1}{2}\right)t\right] \\
&\quad\cdot\cos\left[\left((k_y)_0\,y + \frac{(k_z)_0-(k_z)_1}{2}\right)z - \left(\frac{\omega_0-\omega_1}{2}\right)t\right]
\end{aligned}$$

Irradiance:

$$\begin{aligned}
\left\langle |f[x,y,z,t]|^2\right\rangle &= 4\ \left\langle\cos^2\left[\left(\frac{(k_z)_0+(k_z)_1}{2}\right)z - \left(\frac{\omega_0+\omega_1}{2}\right)t\right]\right\rangle \\
&\quad\cdot\left\langle\cos^2\left[\left((k_y)_0\,y + \frac{(k_z)_0-(k_z)_1}{2}\right)z - \left(\frac{\omega_0-\omega_1}{2}\right)t\right]\right\rangle = 4\cdot\frac{1}{2}\cdot\frac{1}{2} = 1
\end{aligned}$$

No fringes

MORE→→→

3. (15%) Explain the conditions that must exist for a negative lens to create a real image. Repeat for a positive lens that creates a virtual image. Illustrate with sketches.

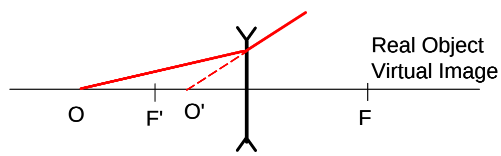
We know that a negative lens creates a virtual image of a real object:

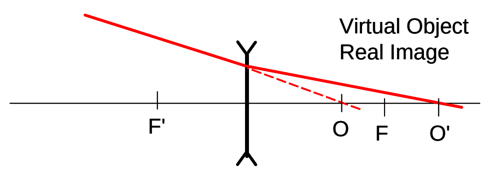
$$s' = \left(\frac{1}{-|f|} - \frac{1}{s}\right)^{-1} < 0 \text{ if } s > 0$$

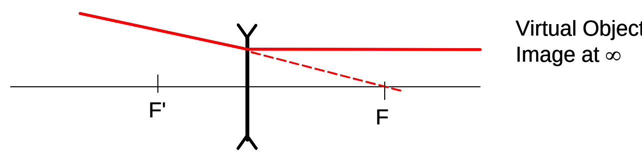
To create a real image, we must have $s' > 0$:

$$0 < s' = \left(\frac{1}{-|f|} - \frac{1}{s}\right)^{-1} \Longrightarrow \frac{1}{-|f|} - \frac{1}{s} > 0 \Longrightarrow \frac{1}{s} < -\frac{1}{|f|} \Longrightarrow s > -|f|$$

The object distance s must be negative (and inside the focal length), so the object must be virtual!



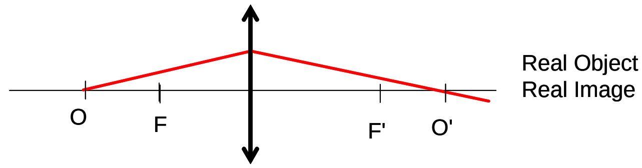


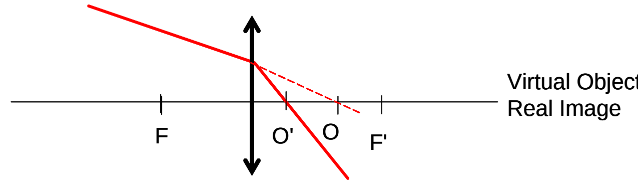


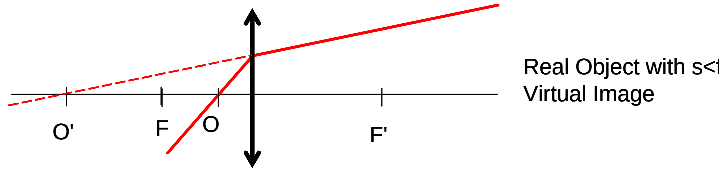
For a positive lens to create a virtual image:

$$s' < 0 = \left(\frac{1}{+|f|} - \frac{1}{s}\right)^{-1} \Longrightarrow \frac{1}{+|f|} - \frac{1}{s} < 0 \Longrightarrow \frac{1}{s} > +\frac{1}{|f|} \Longrightarrow 0 < s < +|f|$$

The object must be real and its distance must be less than the focal length of the lens.

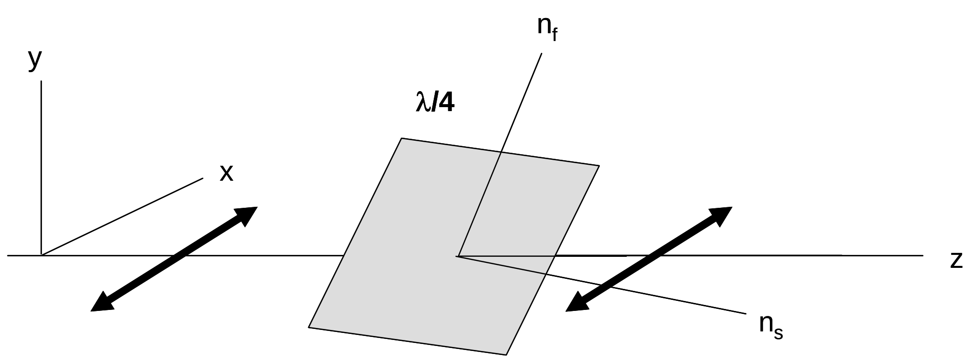






4. (15%) A linearly polarized plane wave travels down the z axis and its electric field is directed along the x axis of the coordinate system. Describe TWO different ways for creating elliptically polarized light from this wave. You may use any of the optical components that we considered in class and/or lab, but specify any critical parameters (e.g., angles of linear polarizers relative to the axes, etc.). Use sketches to illustrate your answer.

   We could make circularly polarized light by introducing a quarter-wave plate with the polarization of the light between the fast and slow axes of the wave plate. If we change the direction of the fast and slow axes, then the resulting light is elliptically polarized UNLESS one of the two axes coincides with the direction of polarization of the incident light. We also could use a waveplate that is not one quarter wave.



5. (10%) Sometimes exams don't cover the aspects of the subject that you studied most carefully or in which you have the greatest interest. For that reason, I offer this "free" question; write your own problem or describe an optical system, effect, or theorem of your choice that we studied this quarter and describe its relevance to imaging. Obviously the phenomena already included in the problems above are excluded for your choice. For example, you could consider the phenomenon of diffraction and its effect on an imaging system. Your score will reflect the suitability of both the question and of the answer.

   See comments on your individual question.