

SIMG-303 SOLUTIONS TO MIDTERM EXAM #1

1. Define the following terms IN WORDS; you may use equations to illustrate your definition, but I'm more interested in the verbal description:

- (a) oscillation (include descriptions of the two necessary forces)

harmonic (and therefore periodic) variation in a physical quantity (e.g., position) created by the interaction of two forces: an "inertial" force of motion and a "restoring" force (often provided by the medium). The functional form of the oscillation is:

$$f [t] = A_0 \cos \left[2\pi \frac{t}{T_0} + \phi \right]$$

- (b) traveling wave

An oscillation or superposition of oscillations that travels through space and time. The argument is a function of both spatial and temporal variables in a linear combination, e.g.,

$$f [z, t] = A_0 \cos \left[2\pi \frac{z}{\lambda_0} \pm 2\pi \frac{t}{T_0} + \phi_0 \right]$$

- (c) index of refraction

The ratio of the velocity of light in vacuum and that in a medium:

$$n = \frac{c}{v_\phi} \geq 1$$

- (d) dispersion relation

The functional relationship of $\omega = \frac{2\pi}{T_0}$ vs. $k = \frac{2\pi}{\lambda_0}$ (i.e., $\omega [k]$) which varies in a medium where waves with different oscillation frequencies travel with different velocities.

- (e) phase velocity

The velocity of a "point of constant phase" on a traveling wave, i.e., the rate of change of position of a point on the wave that was emitted at a specific time. The equation for phase velocity is:

$$v_\phi = \frac{\omega}{k}$$

- (f) modulation velocity (also called "group velocity")

The velocity of a point of constant phase on the so-called "modulation wave", which is the long-period wave created by adding two waves with different periods. In a medium characterized by normal dispersion, the modulation wave travels slower than the average wave. The relevant equation is:

$$\begin{aligned} f_1 [z, t] + f_2 [z, t] &= A_1 \cos [k_1 z + \omega_1 t] + A_1 \cos [k_2 z + \omega_2 t] \\ &= 2A_1 \cos \left[\left(\frac{k_1 + k_2}{2} \right) z + \left(\frac{\omega_1 + \omega_2}{2} \right) t \right] \cdot \cos \left[\left(\frac{k_1 - k_2}{2} \right) z + \left(\frac{\omega_1 - \omega_2}{2} \right) t \right] \\ &= 2A_1 \cos [k_{avg} z + \omega_{avg} t] \cdot \cos [k_{mod} z + \omega_{mod} t] \\ v_{avg} &= \frac{\omega_{avg}}{k_{avg}}, \quad v_{mod} = \frac{\omega_{mod}}{k_{mod}} \end{aligned}$$

2. Given the following complex numbers z_n , evaluate the real part, imaginary part, magnitude, phase angle, z^{-1} and z^* and locate them on an Argand diagram. You may leave quantities in the form of functions – you are not required to evaluate numerical values

(a) $z_1 = 4 \cdot \exp \left[-\frac{3\pi}{4} \right]$

Jeff was right, there was a typographical error, but you received full credit if you solved it as written:

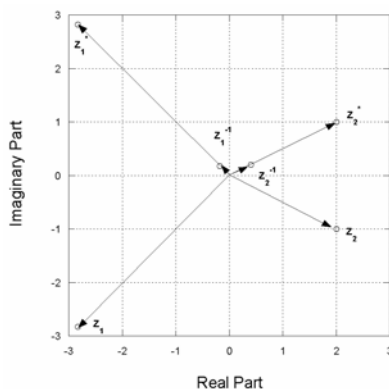
$$\begin{aligned} z_1 &= 4 \cdot \exp \left[-\frac{3\pi}{4} i \right] \\ \operatorname{Re} \{z_1\} &= 4 \cdot \cos \left[-\frac{3\pi}{4} \right] = 4 \cdot -\frac{1}{\sqrt{2}} = -2\sqrt{2} \\ \operatorname{Im} \{z_1\} &= 4 \cdot \sin \left[-\frac{3\pi}{4} \right] = 4 \cdot -\frac{1}{\sqrt{2}} = -2\sqrt{2} \\ |z_1| &= 4 \\ \Phi \{z_1\} &= -\frac{3\pi}{4} = -135^\circ \end{aligned}$$

$$\begin{aligned} z_1^* &= \left(4 \cdot \exp \left[-\frac{3\pi}{4} i \right] \right)^* = 4 \cdot \exp \left[+\frac{3\pi}{4} i \right] \\ &= 4 \cdot \left(\cos \left[+\frac{3\pi}{4} \right] + i \sin \left[+\frac{3\pi}{4} \right] \right) \\ &= -2\sqrt{2} + 2\sqrt{2}i \\ |z_1^*| &= 4 \\ \Phi \{z_1^*\} &= +\frac{3\pi}{4} = +135^\circ \\ z_1^{-1} &= \frac{1}{z_1} = \frac{z_1^*}{|z_1|^2} = \frac{-2\sqrt{2} + 2\sqrt{2}i}{16} = \sqrt{2} \left(\frac{-1 + i}{8} \right) \end{aligned}$$

(b) $z_2 = \frac{6i+3}{3i}$

$$\begin{aligned} z_2 &= \frac{6i+3}{3i} = \frac{6i}{3i} + \frac{3}{3i} = 2 + \frac{1}{i} = 2 + \frac{1}{i} \cdot \frac{-i}{-i} = 2 - i = 2 + (-1)i \\ \operatorname{Re} \{z_2\} &= +2 \\ \operatorname{Im} \{z_2\} &= -1 \\ |z_2| &= \sqrt{2^2 + 1^2} = \sqrt{5} \\ \Phi \{z_2\} &= \tan^{-1} \left\{ \frac{-1}{+2} \right\} \simeq -26.6^\circ \end{aligned}$$

$$\begin{aligned} z_2^* &= 2 + (-1)(-i) = 2 + i \\ z_2^{-1} &= \frac{z_2^*}{|z_2|^2} = \frac{2+i}{5} = \frac{2}{5} + \frac{1}{5}i \end{aligned}$$



(c) $z_3 = \sin \left[\frac{\pi t}{2} \right] + i \cos \left[\frac{\pi t}{2} \right]$

$z_3 = z_3 [t]$: this is a function of t

$$\begin{aligned} z_3 [t] &= \sin \left[\frac{\pi t}{2} \right] + i \cos \left[\frac{\pi t}{2} \right] \\ &= \sin \left[2\pi \frac{t}{4} \right] + i \cos \left[2\pi \frac{t}{4} \right] \\ &= \cos \left[2\pi \frac{t}{4} - \frac{\pi}{2} \right] + i \sin \left[2\pi \frac{t}{4} + \frac{\pi}{2} \right] \\ &= \cos \left[2\pi \frac{t}{4} - \frac{\pi}{2} \right] - i \sin \left[2\pi \frac{t}{4} - \frac{\pi}{2} \right] \\ &= \exp \left[-i \left(2\pi \frac{t}{4} - \frac{\pi}{2} \right) \right] = \exp \left[-i \left(2\pi \frac{t}{4} \right) + i \frac{\pi}{2} \right] \end{aligned}$$

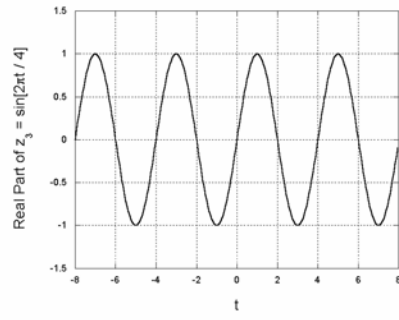
$$|z_3| = +1$$

$$\Phi \{z_3\} = -2\pi \frac{t}{4} + \frac{\pi}{2}$$

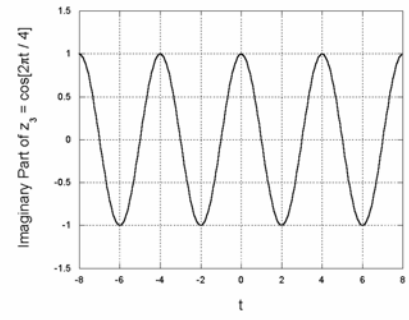
$$z_3^* [t] = \exp \left[+i \left(2\pi \frac{t}{4} - \frac{\pi}{2} \right) \right] = \cos \left[2\pi \frac{t}{4} - \frac{\pi}{2} \right] + i \sin \left[2\pi \frac{t}{4} - \frac{\pi}{2} \right]$$

$$z_3^{-1} = \frac{z_3^*}{|z_3|^2} = \frac{\exp \left[+i \left(2\pi \frac{t}{4} - \frac{\pi}{2} \right) \right]}{1} = z_3^*$$

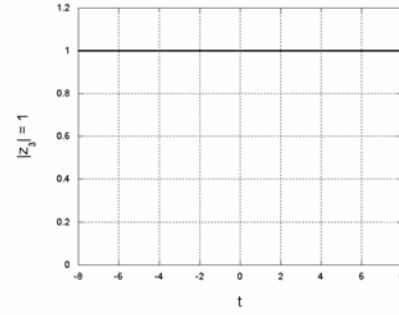
(a)



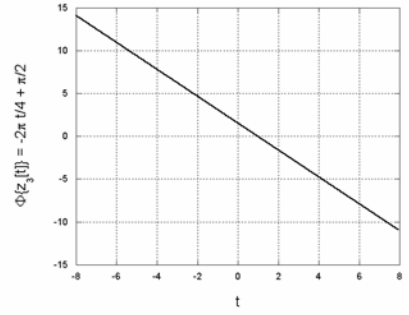
(b)



(c)



(d)



3. Two traveling waves have slightly different temporal frequencies ν_1 and $\nu_2 = \nu_1 + \Delta\nu$, and thus slightly different wavelengths λ_1 and $\lambda_2 = \lambda_1 + \Delta\lambda$. Demonstrate the validity of the expression:

$$\left| \frac{\Delta\lambda}{\lambda} \right| \simeq \left| \frac{\Delta\nu}{\nu} \right|$$

We know the relation for the phase velocity:

$$\begin{aligned} v_\phi &= \frac{\omega}{k} = \nu\lambda \\ d(v_\phi) &= d(\nu\lambda) = \nu \cdot d\lambda + d\nu \cdot \lambda \end{aligned}$$

If the two wavelengths and the two temporal frequencies are only slightly different, then we can say that $d(v_\phi) = 0$

$$\begin{aligned} d(v_\phi) &= 0 = \nu \cdot d\lambda + d\nu \cdot \lambda \\ \implies &\nu \cdot d\lambda = -d\nu \cdot \lambda \\ \implies &\frac{d\lambda}{\lambda} = -\frac{d\nu}{\nu} \\ \implies &\left| \frac{d\lambda}{\lambda} \right| = \left| \frac{d\nu}{\nu} \right| \end{aligned}$$

4. The dispersion relation for sinusoidal waves in the ionosphere is:

$$\omega^2 = \omega_p^2 + c^2 k^2$$

where ω_p is a constant, the so-called *plasma frequency*. A typical approximate value for ω_p is:

$$\omega_p = 2\pi\nu_p \simeq 0.125 \times 10^9 \frac{\text{radians}}{\text{sec}}$$

- (a) Find expressions for the phase velocity v_ϕ and the modulation (or group) velocity v_{mod} . (HINT: evaluate total derivatives of both sides).

$$\begin{aligned} v_\phi &= \frac{\omega}{k} = \frac{\sqrt{\omega_p^2 + c^2 k^2}}{k} \\ &= \sqrt{c^2 + \frac{\omega_p^2}{k^2}} = c \cdot \sqrt{1 + \frac{\omega_p^2}{(ck)^2}} \end{aligned}$$

Take derivatives of both sides

$$\begin{aligned} d(\omega^2) &= d(\omega_p^2 + c^2 k^2) \\ d(\omega^2) &= 2\omega \cdot d\omega \\ d(\omega_p^2 + c^2 k^2) &= d(\omega_p^2) + d(c^2 k^2) \\ &= 2\omega_p \cdot d\omega_p + c^2 d(k^2) \\ &= 2\omega_p \cdot 0 + c^2 d(k^2) \text{ because } \omega_p \text{ is a constant} \\ &= c^2 \cdot 2k \cdot dk \end{aligned}$$

Equate left- and right-hand sides:

$$2\omega \cdot d\omega = c^2 \cdot 2k \cdot dk$$

$$\frac{\omega}{k} = v_{avg} = c^2 \left(\frac{d\omega}{dk} \right)^{-1} = \frac{c^2}{v_{mod}}$$

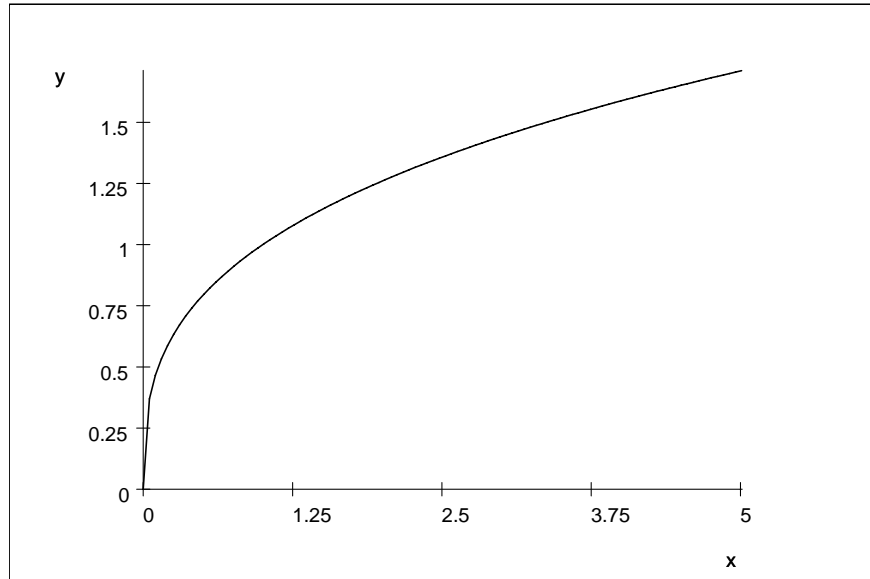
Calculate $v_{\phi} \cdot v_{mod}$

$$\frac{\omega}{k} \cdot \frac{d\omega}{dk} = c^2 \implies v_{avg} \cdot v_{mod} = c^2$$

5. (Read carefully) The phase velocity of the waves in some medium is proportional to ω^{-2}

(a) Sketch the dispersion relation $\omega [k]$ in this medium.

$$v_{\phi} = \frac{\omega}{k} \propto \omega^{-2} \implies \omega^3 = k \implies \omega = k^{\frac{1}{3}}$$



$\omega = k^{\frac{1}{3}}$, where k on the horizontal (x) axis and ω is plotted on the vertical (y) axis.

(b) Find an expression for the modulation velocity

$$v_{mod} = \frac{d\omega}{dk} = \frac{d}{dk} \left(k^{\frac{1}{3}} \right) = \frac{1}{3} k^{-\frac{2}{3}}$$

(c) Do the waves exhibit normal or anomalous dispersion?

$$v_{avg} = \frac{\omega}{k} = \frac{\left(k^{\frac{1}{3}} \right)}{k} = k^{-\frac{2}{3}} > \frac{1}{3} k^{-\frac{2}{3}}$$

So $v_{avg} > v_{mod}$ and the dispersion is normal

6. Consider the superposition of two sinusoidal traveling waves:

$$\begin{aligned} f_1 [z, t] &= A_1 \sin [k_1 z - \omega_1 t] \\ f_2 [z, t] &= -A_2 \sin [k_2 z - \omega_2 t] \end{aligned}$$

where $A_1 = A_2 = 10 \text{ mm}$, $\nu_1 = 100 \text{ Hz}$, $v_1 = 2.5 \frac{\text{m}}{\text{s}}$, $\nu_2 = 150 \text{ Hz}$, $v_2 = 5.0 \frac{\text{m}}{\text{s}}$

(a) Find the resulting wave in terms of the average wave and the modulation wave.

$$\begin{aligned} f_1 [z, t] + f_2 [z, t] &= 10 (\sin [k_1 z - \omega_1 t] - \sin [k_2 z - \omega_2 t]) \\ &= 10 \left(\cos \left[k_1 z - \omega_1 t - \frac{\pi}{2} \right] - \cos \left[k_2 z - \omega_2 t - \frac{\pi}{2} \right] \right) \\ &= 10 \left(\cos \left[k_1 z - \omega_1 t - \frac{\pi}{2} \right] + \cos \left[k_2 z - \omega_2 t + \frac{\pi}{2} \right] \right) \\ \phi_1 &= -\frac{\pi}{2}, \phi_2 = +\frac{\pi}{2}, \phi_{avg} = \frac{\phi_1 + \phi_2}{2} = 0, \phi_{mod} = \frac{\phi_1 - \phi_2}{2} = -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} f_1 [z, t] + f_2 [z, t] &= 20 \cdot \cos [k_{avg} z - \omega_{avg} t + \phi_{avg}] \cdot \cos [k_{mod} z - \omega_{mod} t + \phi_{mod}] \\ &= 20 \cdot \cos [k_{avg} z - \omega_{avg} t + \phi_{avg}] \cdot \cos [k_{mod} z - \omega_{mod} t + \phi_{mod}] \\ &= 20 \cdot \cos [k_{avg} z - \omega_{avg} t] \cdot \cos \left[k_{mod} z - \omega_{mod} t - \frac{\pi}{2} \right] \end{aligned}$$

(b) Calculate the wavelengths of the two waves f_1 and f_2 .

$$\begin{aligned} \lambda_1 \cdot \nu_1 &= v_1 \implies \lambda_1 = \frac{v_1}{\nu_1} = \frac{2.5 \frac{\text{m}}{\text{s}}}{100 \text{ Hz}} = 2.5 \cdot 10^{-2} \text{ m} = 25 \text{ mm} \\ \lambda_2 \cdot \nu_2 &= v_2 \implies \lambda_2 = \frac{v_2}{\nu_2} = \frac{5.0 \frac{\text{m}}{\text{s}}}{150 \text{ Hz}} \simeq 3.33 \cdot 10^{-2} \text{ m} = 33.3 \text{ mm} \end{aligned}$$

(c) Find the velocity of the average wave and of the modulation wave.

$$\begin{aligned} v_{avg} &= \frac{\omega_{avg}}{k_{avg}} = \frac{\frac{2\pi}{2} (\nu_1 + \nu_2)}{\frac{2\pi}{2} \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)} = \frac{(250 \text{ Hz})}{\frac{1}{25 \text{ mm}} + \frac{1}{33.3 \text{ mm}}} \approx 3.57 \frac{\text{m}}{\text{s}} \\ v_{mod} &= \frac{\omega_{mod}}{k_{mod}} = \frac{\frac{2\pi}{2} (\nu_1 - \nu_2)}{\frac{2\pi}{2} \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right)} = \frac{(-50 \text{ Hz})}{\frac{1}{25 \text{ mm}} - \frac{1}{33.3 \text{ mm}}} \approx -5 \frac{\text{m}}{\text{s}} \end{aligned}$$

(d) Does this system exhibit normal or anomalous dispersion?

The shorter wavelength wave #1 travels slower, so the dispersion is normal

7. We derived an expression for the superposition (sum) of multiple oscillations with the same frequency. Convince me with words and diagrams that this result is correct.

The sum of two oscillations with the same frequency had the form

$$f_1 [t] + f_2 [t] = A_1 \cos [\omega_1 t + \phi_1] + A_2 \cos [\omega_1 t + \phi_2]$$

which can be written as the real part of the sum of two complex waves:

$$f_1 [t] + f_2 [t] = \text{Re} \{A_1 \exp [i (\omega_1 t + \phi_1)] + A_2 \exp [i (\omega_1 t + \phi_2)]\}$$

Consider the complex function:

$$\begin{aligned} f_1 [t] + f_2 [t] &= A_1 \exp [i (\omega_1 t + \phi_1)] + A_2 \exp [i (\omega_1 t + \phi_2)] \\ &= A_1 \exp [i \omega_1 t] \exp [i \phi_1] + A_2 \exp [i \omega_1 t] \exp [i \phi_2] \\ &= (A_1 \exp [i \phi_1] + A_2 \exp [i \phi_2]) \exp [i \omega_1 t] \end{aligned}$$

which rotates around the origin at the constant frequency ω_1 , hence the frequency of the sum is the same as of the individual terms.

Histogram of Scores:

